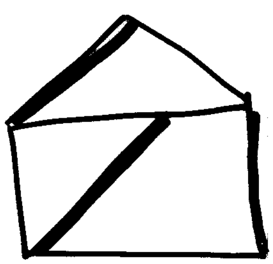
Dimer models: The limit shape phenomenon

(I) A dimer configuration on a graph =

= perfect matching between vertices

= coloring of edges s.t.

- none of the two colored edges share a vertex.
- each vertex belongs to a colored edge



colored edges ≡

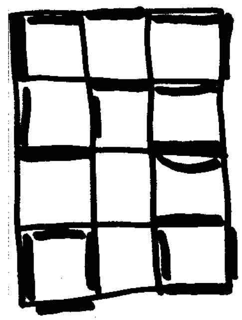
≡ dimers ≡

≡ dimer edges

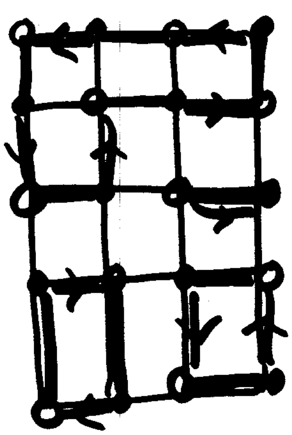
A bipartite graph = vertices are of two types (black and white) edges connect only vertices of different type.

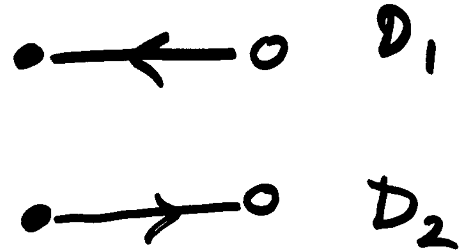
A pair of dimer configurations ⟹

⟹ a collection of nonintersecting, nonselfintersecting loops on a graph



On a bipartite graph an (ordered) pair of dimer configurations ⟹ oriented nonintersecting, nonselfintersecting loops

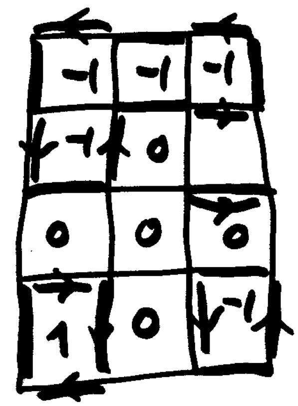




Γ - bipartite planar graph (on an oriented surface).

Given $D_1, D_2, x_0 \in F(\Gamma)$, the height function

$$h : F(\Gamma) \to \mathbb{Z}$$



<u>Weights</u> :

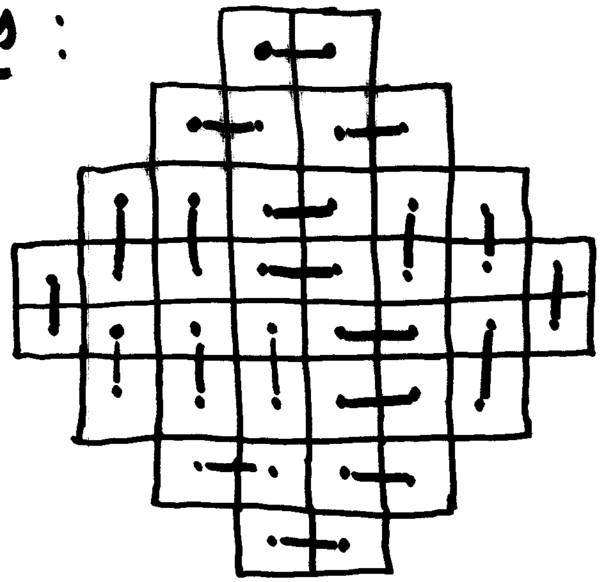
$$w(D) = \prod_{\substack{\text{dimer} \\ \text{edges } e}} w(e) \prod_{\substack{\text{faces} \\ (2\text{ cells})}} q_x^{\,h_{D,D_0}(x,x_0)}$$

D_0 = "ground dimer configuration"

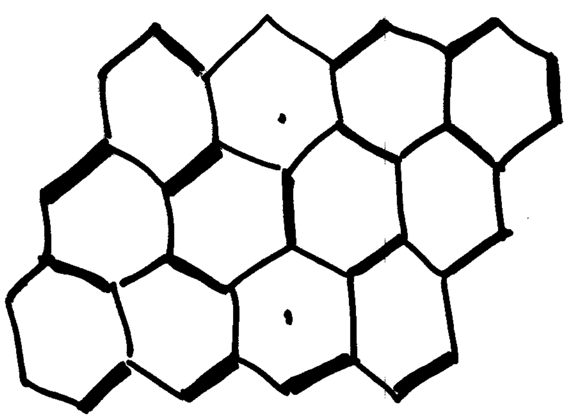
$x_0 \in F(\Gamma)$ = "base point" in $F(\Gamma)$.

Remark: for planar Γ $\{q_x\} \simeq$ change of $w(e)$

Examples :



- tilings of a plane by domino
- count alternating sign matrices

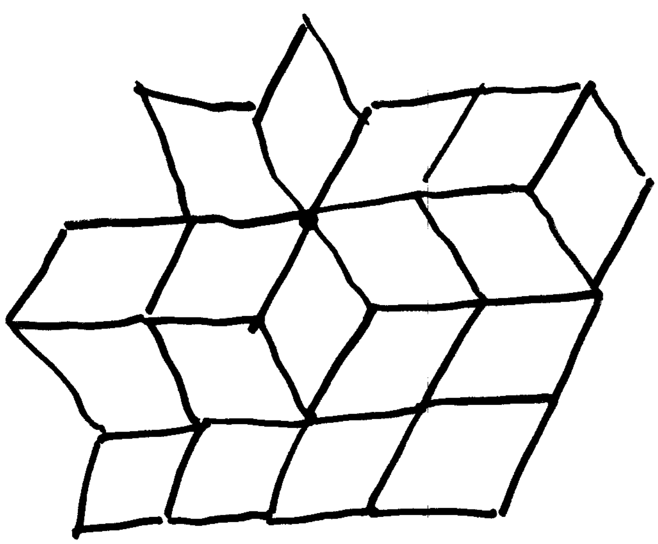


- tilings by rhombi
- 3d partitions

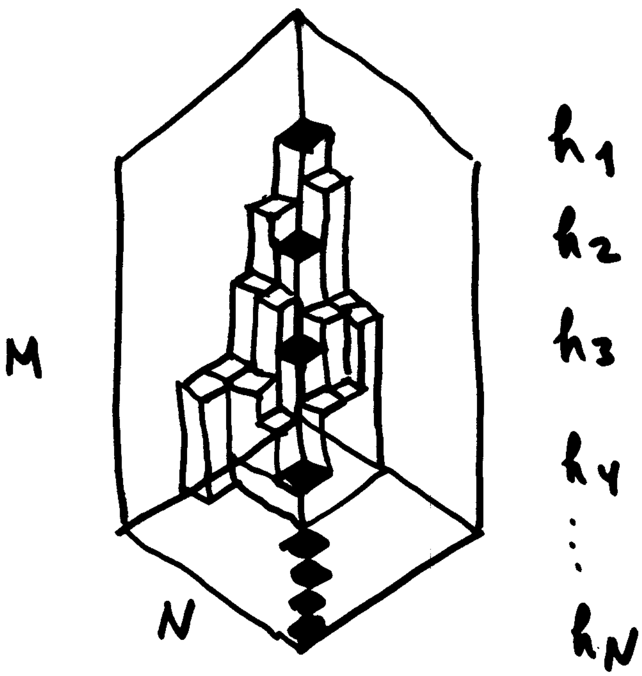
height function = 3d height

if $w(e)=1$, $q_x = q$,

$$w(\text{dimer}) = w(\text{3d partition}) = q^{\mathrm{vol}(\text{3d part})}$$



Relation to matrix integrals:



$$M \geq h_1 > \cdots > h_N \geq -N$$

$$Z_{M,N} = \sum_{\text{dimers}} \prod_{\substack{\text{central}\\ \text{horizontal}\\ \text{tiles}}} q_h^h$$

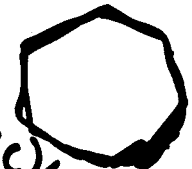
$$= \sum_{N \geq h_1 > \cdots > h_N \geq -N} x_{h_1} \cdots x_{h_n} \left(\sum_{\mathcal{D}_+(h)} \right) \left(\sum_{\mathcal{D}_-(h)} \right) =$$

$$= \sum_{M \geq h_1 > \cdots > h_N \geq N} x_{h_1} \cdots x_{h_n} \left(\prod_{i>j} \frac{h_i - h_j}{i-j} \right)^2 ,$$

$$\int_{\substack{\text{Hermitian}\\ \text{matrices}}} e^{\operatorname{tr}(V(M))} \, dM = \int_{\lambda_1 > \cdots > \lambda_N} \prod_{i=1}^{N} e^{V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d\lambda$$

II The "solution" of dimer models.

- Γ = bipartite and planar
- $\varepsilon : E(\Gamma) \longrightarrow \pm 1$,



C

$$\prod_{e \in C} \varepsilon(e) = (-1)^{\ell(C)/2}$$

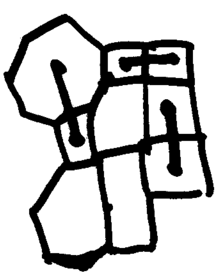
- Define

$$K : \mathbb{C}^{b\text{-vertices}} \longrightarrow \mathbb{C}^{w\text{-vertices}}$$

$$K x_v = \sum_{e : v \to w} \varepsilon(e) w(e) x_w$$

Kasteleyn

- $$\sigma_e(D) = \begin{cases} 1 & , e \in D \\ 0 & , e \notin D \end{cases}$$



$$\varepsilon(i,j) = (-1)^{\#(\text{pathes through } (i,j))}$$

$$\sigma_i \, \varepsilon(i,j) \, \sigma(j)^{-1} \qquad \sigma(i) = \pm 1$$

<u>Thm</u> (Kasteleyn)

1. $$Z_{\text{dimer}} = \det(K)$$

2. $$\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \det_{i,j=1\dots n} \left(K^{-1}_{b_i, w_j} \right)$$

If K is sufficiently simple, one can compute K, K^{-1}, ..., Ising model, free fermionic point of 6v-model, ... (Kasteleyn, Fisher, McCoy, Wu, Lieb, ...)

Most interesting phenomena, including phase transitions, at $|\Gamma| \to \infty$.

III The limit shape phenomenon

$$w(D) = \prod_{e \in D} w(e) \cdot q^{\sum_x h(x)} = \prod_{e \in D} w(e) \cdot q^{vol}$$

as $|\Gamma| \to \infty$ there are two possibilit.

($w(e) = 1$)

- $q < 1$, the partition function remains finite
- $q = e^{-\frac{\varepsilon}{L}}$, L = characteristic size of Γ, $L \to \infty$

  in this limit

  $$Z \propto \exp(f \cdot L^2)$$

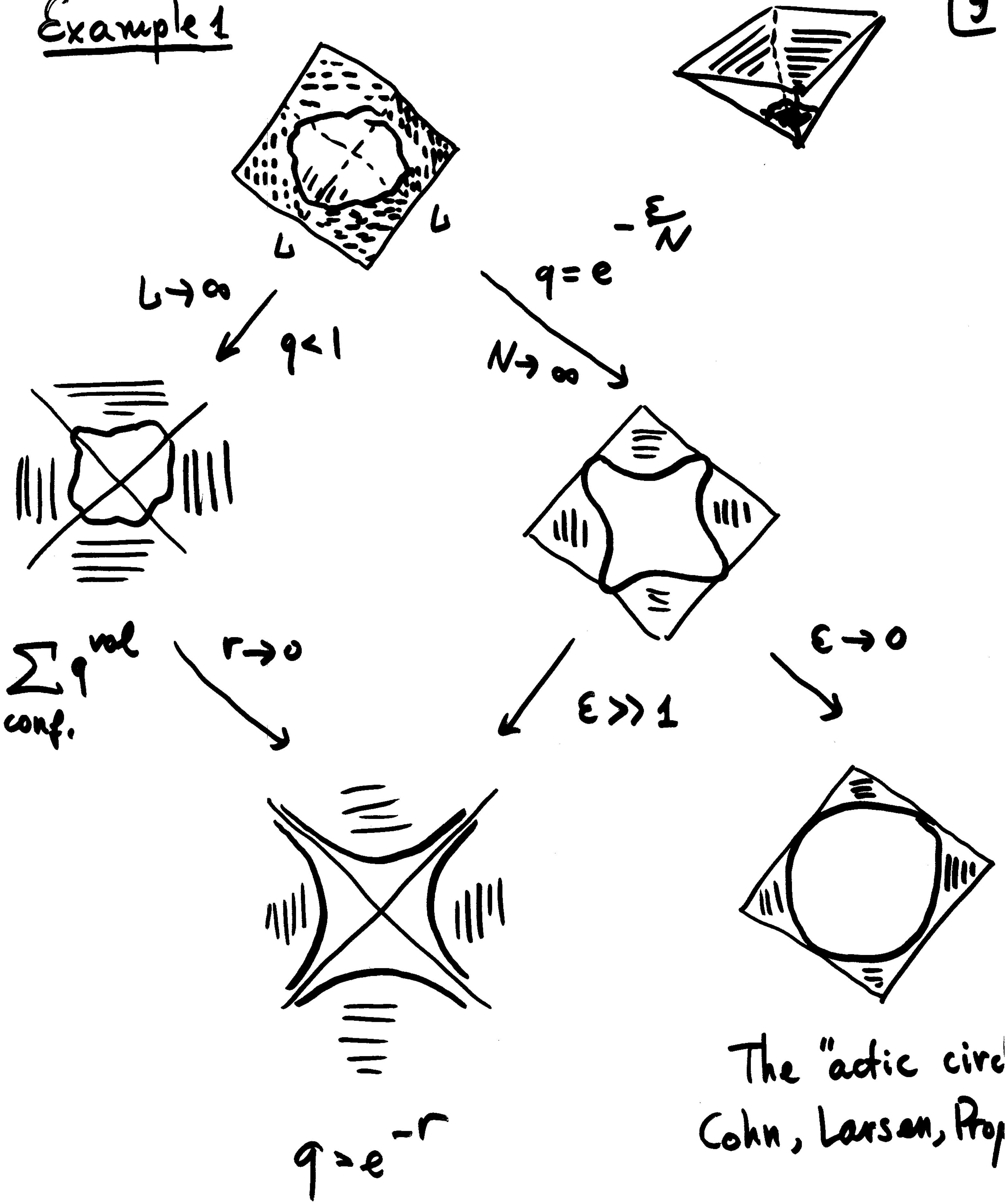
  and

one configuration D_{lim} "becomes dominant" : as $L \to \infty$

$$\text{probab}(D) \propto \exp(-L\, d(D, D_{lim}))$$

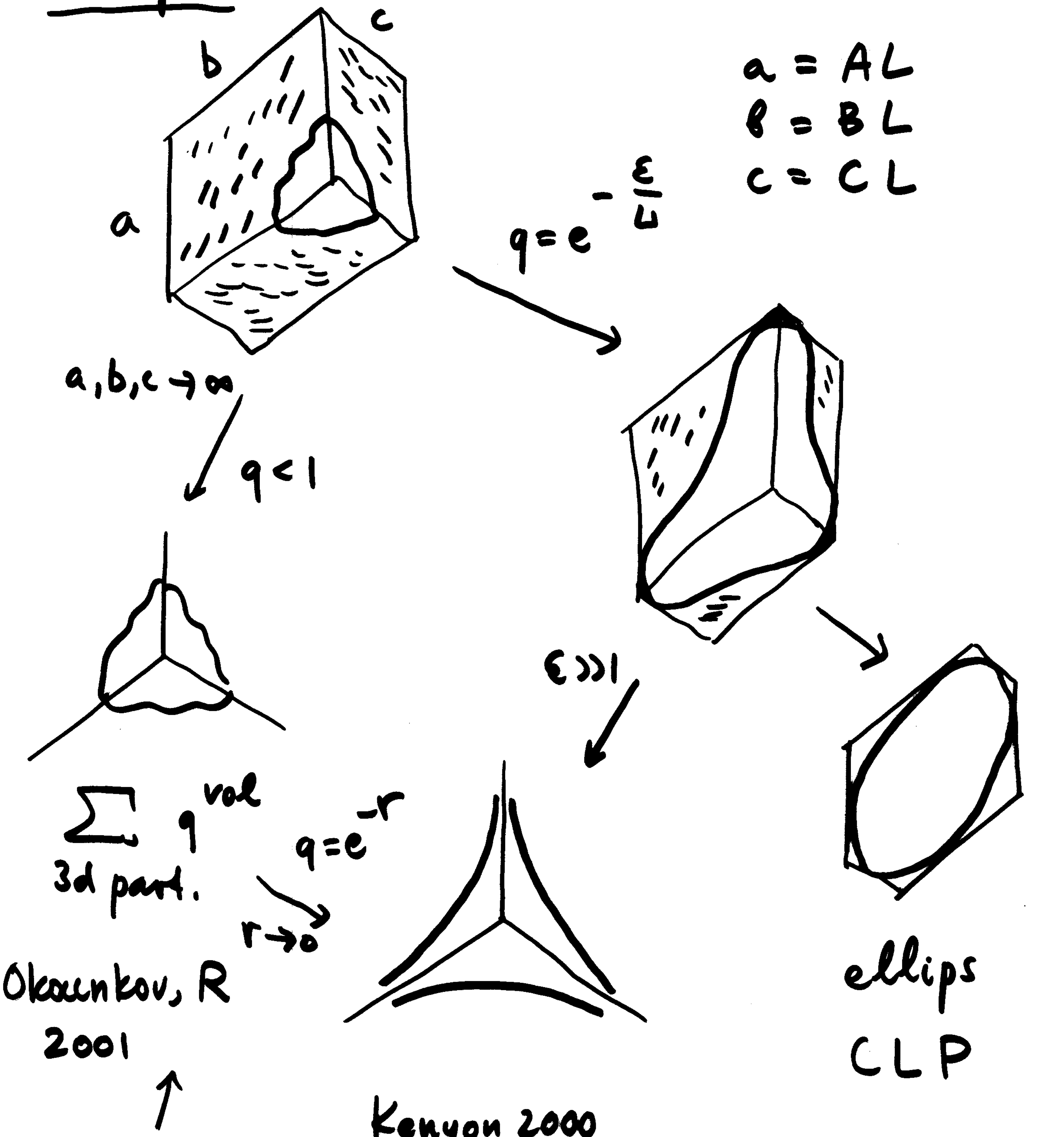
D_{lim} = The limit shape

Example 1



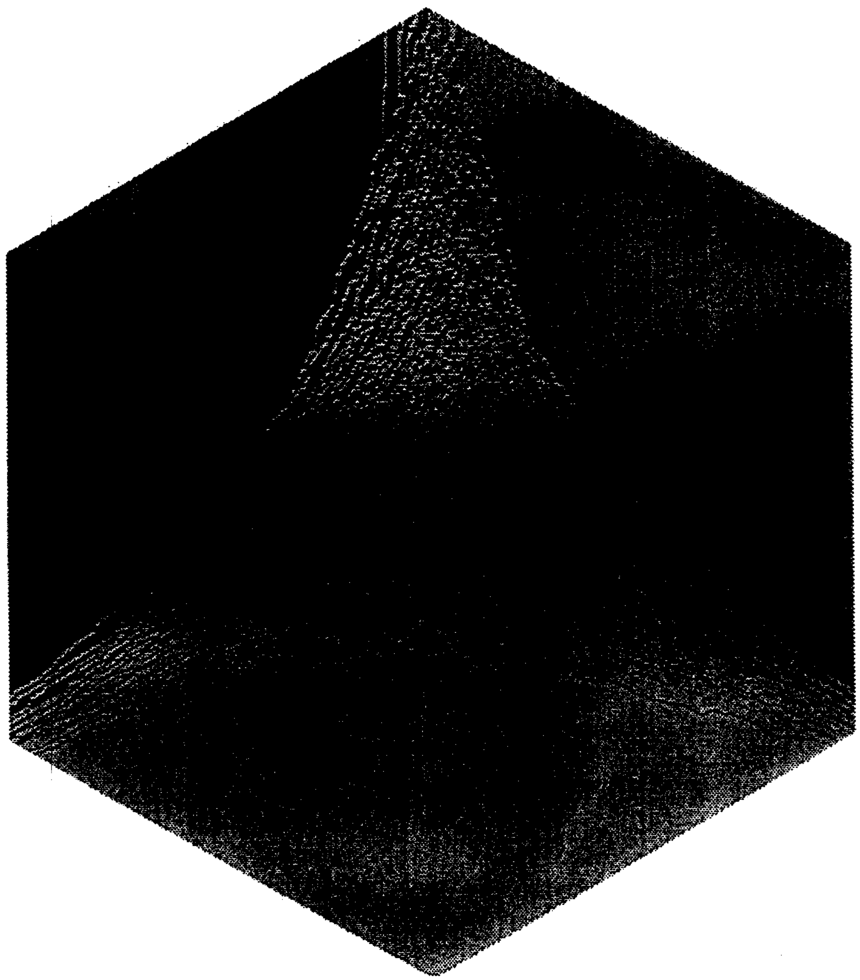
The "actic circ
Cohn, Larsen, Prop

Example 2



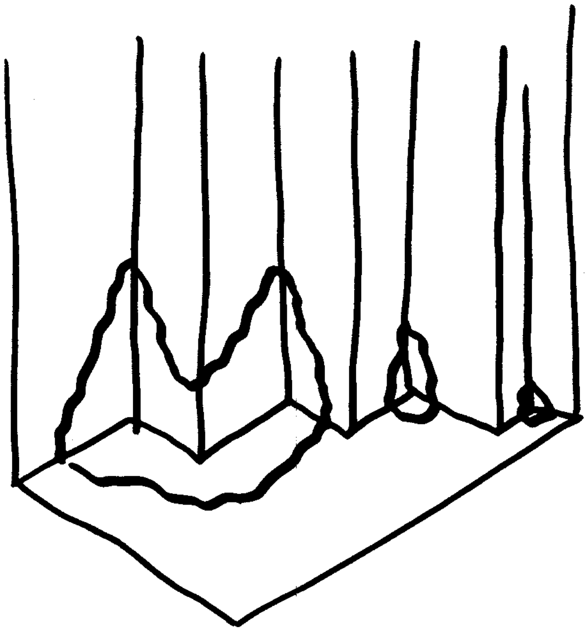
K is "simple"

K^{-1} can be expl. computed



$q<1, \quad q^{vol}$

Example 3 "Solvable" (Okounkov, R, ...)



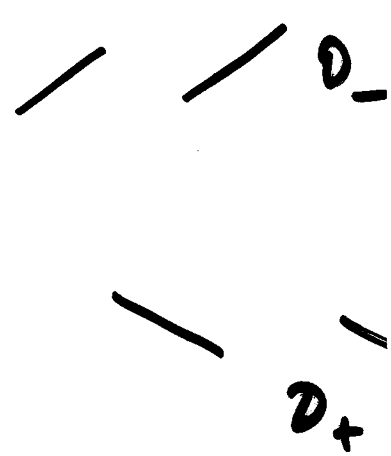
$$q^{\text{vol}}, \quad q<1$$

$$K^{-1}_{(t_1,h_1),(t_2,h_2)} = \frac{1}{(2\pi i)^2} \int_{C_1}\int_{C_2} \frac{\Phi_-(z,t_1)\,\Phi_+(w,t_2)}{\Phi_+(z,t_1)\,\Phi_-(w,t_2)}$$

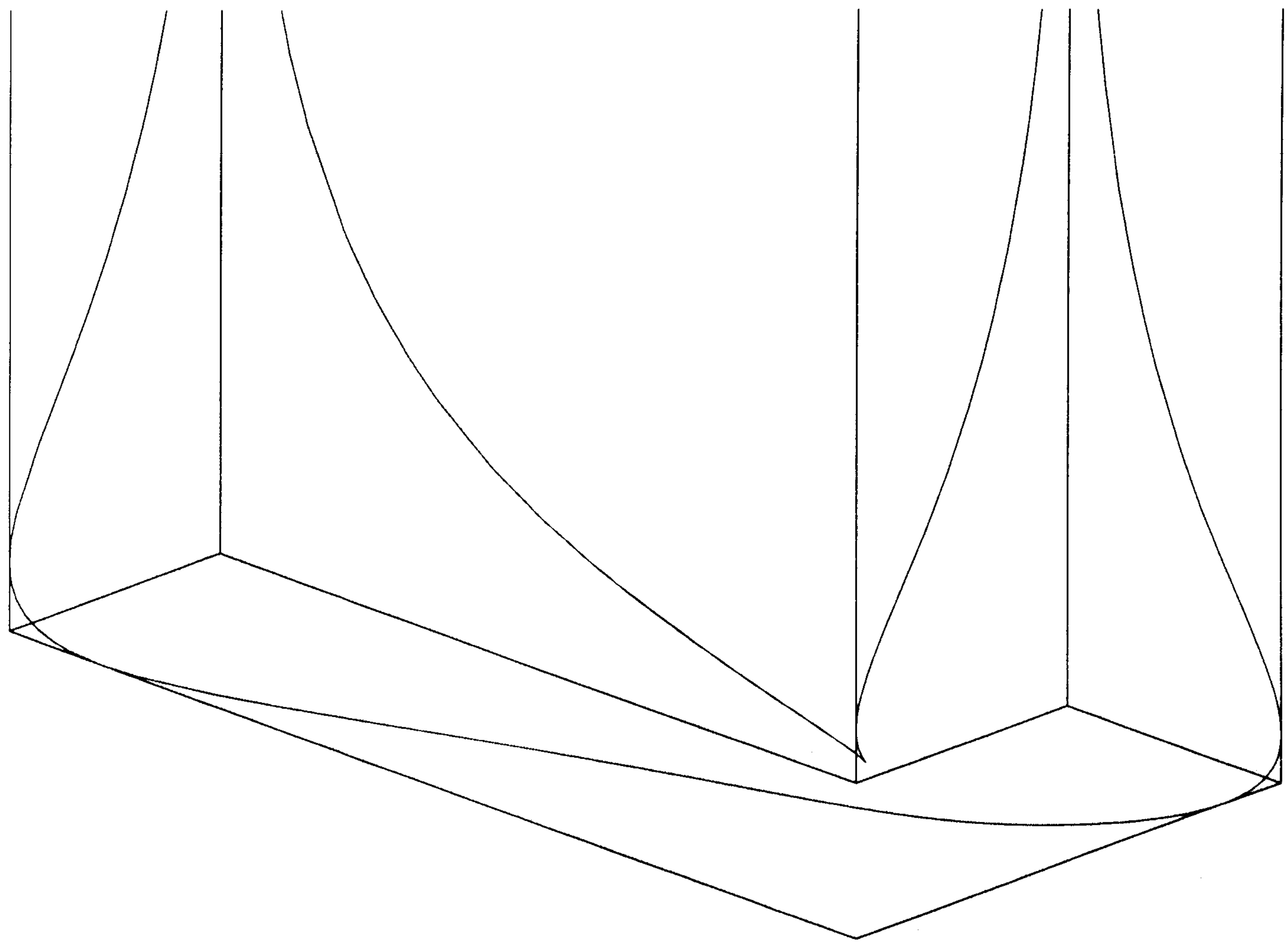
$$\frac{\sqrt{zw}}{z-w}\, z^{-j_1} w^{j_2} \;\frac{dz\,dw}{zw},$$

$$\Phi_+(z,t) = \prod_{\substack{m>t \\ m\in D_+}} (1 - z q^m)$$

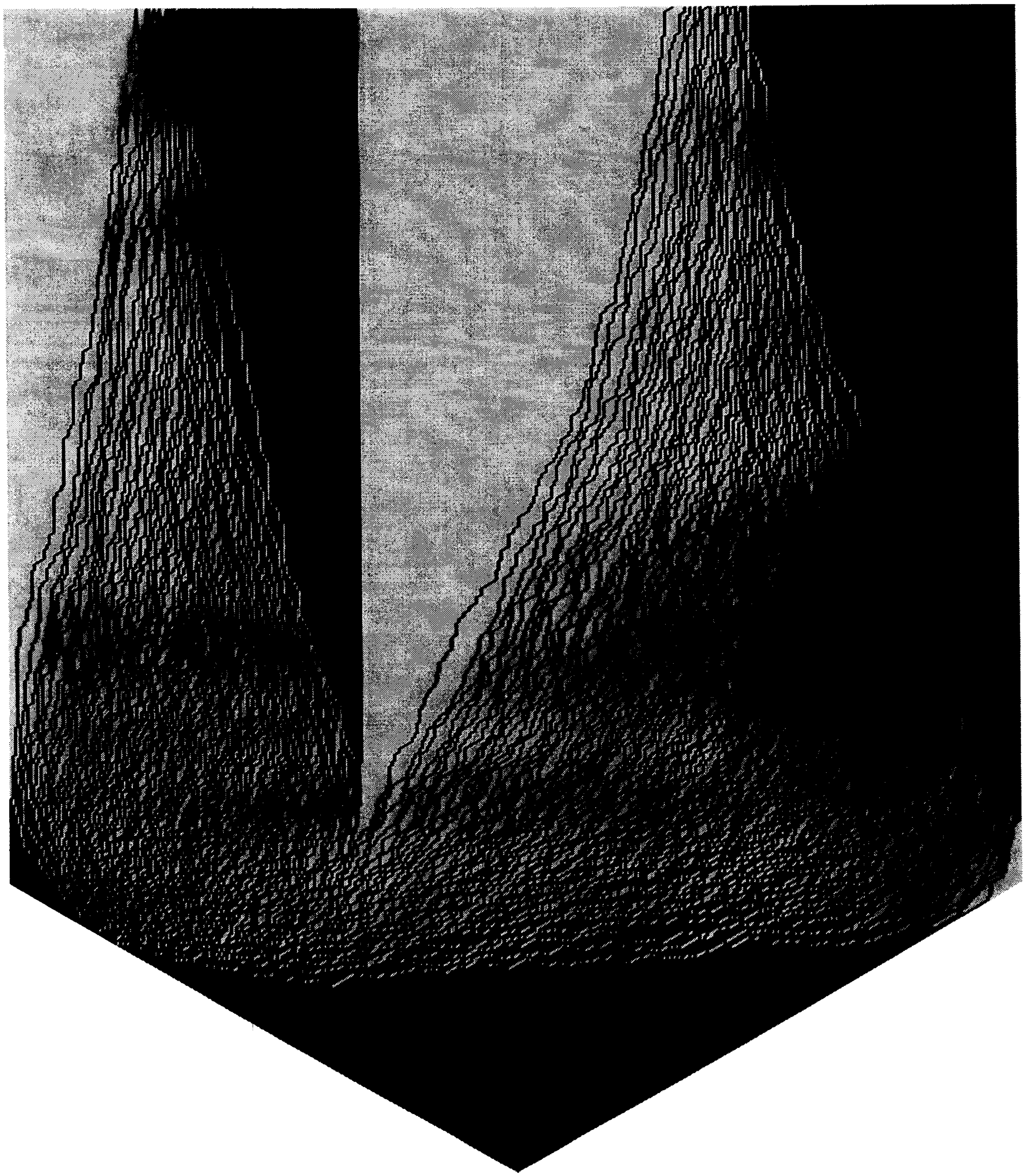
$$\Phi_-(z,t) = \prod_{\substack{m<t \\ m\in D_-}} (1 - z^{-1} q^{-m})$$



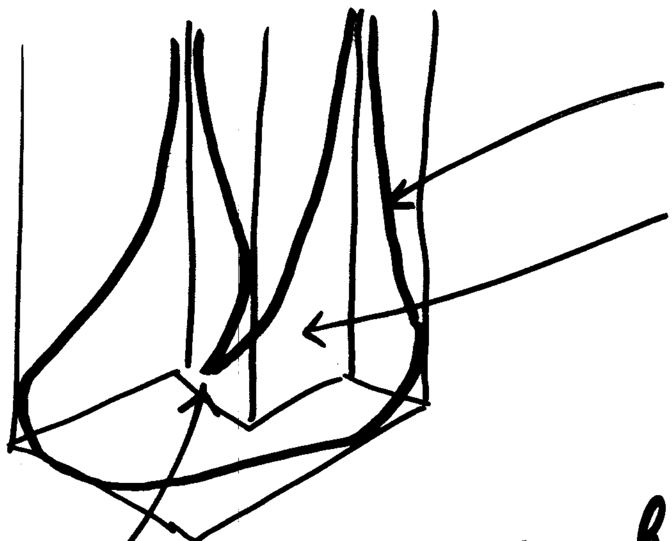
Limit shape for skew 3D Y.d. (11.1



- Peasey, near the spike

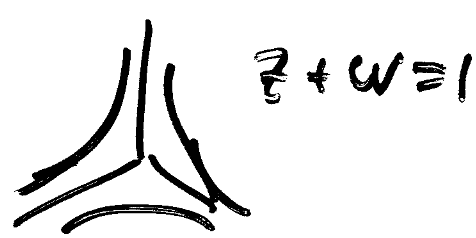


The asymtotic of correlation functions



- bulk:
  e.g. <u>discrete sin kernel</u>
  (OR, 2001)
- edge scaling, boundary:
  double Airy scaling
  (Ferrary, Sphon, 2002)
- cusp: Pearsey scaling, integral
  triple (Okounkov, R, ...)

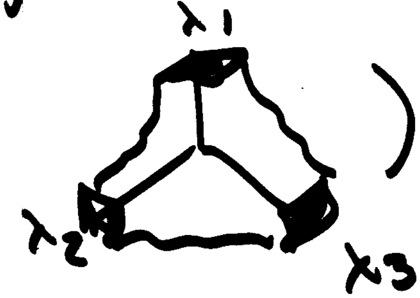
$$\int e^{-x^4} \ldots$$



Some important developments:

- Γ = double periodic graph
  $w(e)$ – double periodic, $q<1$,
  as $q \to 1$ the limit shap is the
  amoeba of the spectral curve
  $$P(z,w) = \det(K(z,w))$$
  Kenyon, Okounkov, Sheffield

- limit shapes $\leftrightarrow$ Harnack curves
  $\updownarrow$
  [double periodic weights] $\nearrow$
  Kenyon, Okounkov

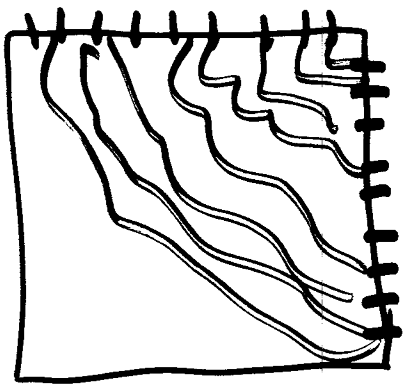
- $Z($ 

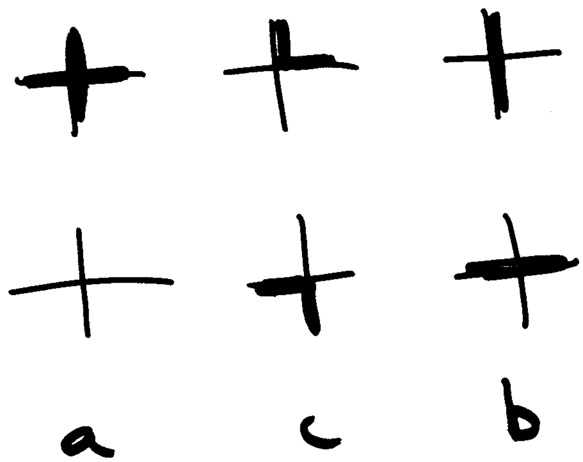
  $)(q)$ = the topological vertex

  Okounkov, R., Vafa

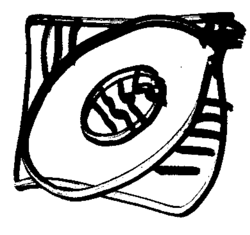
- $\exp(GW(h)) = DT(-e^{ih})$
  Maulik, Nekrasov, Okounkov, Pandharipande

(IV) Beyond dimers.
6-vertex model





$$\Delta = \frac{a^2+b^2-c^2}{2ab},$$



$\Delta = 0$ free fermionic point

$(6v., \Delta=0) \simeq$ (Aztek diamond, dominoe, alt. sign. m.)

limit shape $\Leftarrow$ limit shape

On going work with K. Palamarchuk & D. Allison

6v., Domain Wall boundary conditions develop limit shape

More general phenomenon, Glauber d.