Celebrating

50 years of contributions

from Albert Schwarz

Morita equivalence

for non-commutative tori

Marc A. Rieffel

1972 Slawny: Non-comm. tori.

Deformation quantizations of tori.

Fourier picture.

$C(T^d) \cong C^*(Z^d)$, convolution.

Let Θ be $d \times d$ skew matrix / $\mathbb{R}$.

For $f, g \in \ell^1(Z^d)$, $m, n \in Z^d$

$$(f \underset{\Theta}{*} g)(n) = \sum_m f(m) g(n-m) e^{\pi i (\Theta n) \cdot m}.$$

Acts on $\ell^2(Z^d)$, same formula.

Complete for operator norm.

Denote by T_Θ. Unitary generators:

U_n: $\quad U_m U_n = e^{2\pi i (\Theta n) \cdot m} U_n U_m$.

<u>Def</u>: a left A-module V is (f.g.) <u>projective</u> if V is a direct summand of $({}_A A)^n$.

Is an analog of vector bundles (Swan)

<u>Def</u>: Two unital algebras, A and B, are <u>Morita equivalent</u> $(A \overset{M}{\simeq} B)$ if their module categories are $\simeq$.

<u>Theorem (Morita)</u>: $A \overset{M}{\simeq} B$ iff there is a bimodule ${}_A\Xi_B$ such that

$$ {}_B V \longmapsto {}_A\Xi \otimes_B V $$

is category equivalence, for which need $A = \mathrm{End}_B(\Xi_B)$, and Ξ_B is projective, generator.

E.g.: T_θ, $d=2$, $\theta \in \mathbb{R}$, $\theta \neq 0$.
Take Fourier xform in one variable, get $C(\mathbb{T})$. Other variable U_n, $n \in \mathbb{Z}$.

$$U_n f = \alpha_n(f) U_n, \quad (\alpha_n(f))(t) = f(t - n\theta).$$

$$T_\theta = C(\mathbb{T}) \times_\alpha \mathbb{Z}.$$

Let $\Xi = C_c(\mathbb{R})$.

$C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z})$ acts pointwise.

$$(U_n \xi)(t) = \xi(t - n\theta), \quad \xi \in \Xi.$$

Makes Ξ a T_θ-module, projective, not free.

$\mathrm{End}_{T_\theta}(\Xi)$ generated by

$C(\mathbb{R}/\theta\mathbb{Z})$ and $(V_n\xi)(t) = \xi(t-n)$.

Gives right action of $T_{(1/\theta)}$,

and $T_\theta \overset{M}{\cong} T_{(1/\theta)}$.

For $k \in \mathbb{Z}$, $T_{\theta+k} = T_\theta$

but $T_{(1/(\theta+k))} \neq T_{(1/\theta)}$

<u>Theorem (R. 1980)</u>: For $d=2$,

$T_{\theta_1} \overset{M}{\cong} T_{\theta_2}$ iff θ_1 and θ_2

are in same $SL(2,\mathbb{Z})$-orbit,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\theta = \frac{a\theta+b}{c\theta+d}.$$

Use Ξ, or more generally

$\Xi = C_c(\mathbb{R}\times\mathbb{Z}_q)$, suitably completed.

1980 Connes: First paper on non-commutative differential geometry.

For action α of Lie group G on a C^*-algebra A, have set A^∞ of smooth vectors, a dense $*$-subalgebra. Have bijection between projective A-modules and A^∞-modules.

E.g. T_θ. Have action α of $G = T^d$ by

$$(\alpha_t(f))(n) = e^{2\pi i n \cdot t} f(n).$$

Then $T_\theta^\infty = \mathcal{S}(\mathbb{Z}^d)$.

Smooth Ξ is $\mathcal{S}(\mathbb{R})$ $(d=2)$.

<u>1980 Connes</u> (cont.): On projective A^∞-module define <u>connections</u>,

$$\nabla : \mathfrak{g} \longrightarrow \mathrm{Lin}(\Xi),$$

$$\nabla_X(a\xi) = \alpha_X(a)\xi + a\nabla_X(\xi).$$

Get curvature, Chern classes.

<u>1987 Connes + R.</u>: Define Yang-Mills functional on space of connections on projective A^∞-module; define gauge group and its action on connections.

For T_θ^∞, $d=2$, determined moduli space of minima of YM (instantons?). It is T^2 or modifications, depending on Ξ.

<u>1988 R.</u>: For all d construct projective T^∞_θ-modules generalizing $\mathcal{S}(\mathbb{R}\times\mathbb{Z}_q)$. Use $M=\mathbb{R}^a\times\mathbb{Z}^b\times F$ for F a finite Abelian group. Have T^∞_θ-module action on $\mathcal{S}(M)$ via $\mathbb{Z}^d \hookrightarrow M\times\hat{M}$ as lattice, $(M\times\hat{M})/\mathbb{Z}^d$ compact. Call $\Xi=\mathcal{S}(M)$ a <u>"Heisenberg" module</u>. Each has a connection of <u>constant curvature</u>:

$$F(X,Y)=[\nabla_X,\nabla_Y]=\omega(X,Y)\,I_E\,,$$

$X,Y\in\mathfrak{g}=\mathbb{R}^d$, $E=\mathrm{End}_{T^\infty_\theta}(\Xi)$.

Classify Ξ by Chern character.

<u>Theorem.</u> If θ not rational, then every projective T^∞_θ-module is $\oplus$ of a finite number of Heisenberg modules;

1997 Connes, Douglas, Schwarz:

T_θ^∞ and connections on projective T_θ^∞-modules arise naturally in string theory, for "compactifications". In BFSS and IKKT matrix models, conjectured to converge to the conjectured M-theory overarching string theory, one has self-adjoint matrices X_j, $j = 1, \cdots, d$, analogs of coordinates on $\mathbb{R}^d$.

To "compactify" the X_j's to analogs of circles of radius r_j, one wants each X_j unitarily equivalent to $X_j + 2\pi i r_j I$, i.e. unitaries U_j so

$$U_j X_j U_j^* = X_j + 2\pi i r_j I, \quad j = 1, \cdots, d.$$

The U_j's need not commute, but find $U_j U_k U_j^* U_k^*$ commutes with all, so best to set $U_j U_k U_j^* U_k^* = e^{2\pi i \theta_{jk}} I$. Thus the U_j's generate T_Θ^∞. The X_j's can then be realized as connections on projective T_Θ^∞-modules.

1998 R. and Schwarz:

Morita equivalence for T_θ's, $d > 2$.

Let $O(d, d; \mathbb{R})$ be the group preserving the form from $\begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$.

For $g \in O(d, d; \mathbb{R})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ set

$$g\theta = (a\theta + b)(c\theta + d)^{-1}$$

(if exists). Say that θ is "<u>generic</u>" if $(c\theta + d)^{-1}$ exists for all g in $\underline{S}O(d, d; \mathbb{Z})$.

<u>Theorem</u>. If θ is generic, then

$$T^\infty_{g\theta} \overset{M}{\simeq} T^\infty_\theta \text{ and } T_{g\theta} \overset{M}{\simeq} T_\theta$$

for all $g \in SO(d, d; \mathbb{Z})$.

Proof is by constructing Heisenberg modules for generators of $SO(d,d;\mathbb{Z})$.

But for $d > 2$ there exist θ_1 and θ_2 with $T_{\theta_1} \cong T_{\theta_2}$ but θ_1 and θ_2 <u>not</u> in same $SO(d,d;\mathbb{Z})$-orbit, and $T^\infty_{\theta_1} \ncong T^\infty_{\theta_2}$! So we can have distinct smooth structures on T_θ already for $d = 3$.

The classification of T_θ's up to $\cong$ is not fully known.

1998 Schwarz: Observed that for a Heisenberg module Ξ which gives $T^\infty_{\theta_1} \overset{M}{\simeq} T^\infty_{\theta_2}$, the natural connection, ∇^Ξ, is simultaneously (after suitable) identifications, both a $T^\infty_{\theta_1}$ and a $T^\infty_{\theta_2}$ connection, both with constant curvature. Schwarz calls this "gauge Morita equivalence".

Schwarz shows that its importance is:

∇^{Ξ} determines a bijection between the connections on a projective $T^{\infty}_{\Theta_2}$-module V and the connections on the $T^{\infty}_{\Theta_1}$-module $W = \Xi \otimes_{T^{\infty}_{\Theta_2}} V$, given by

$$\nabla_X \longmapsto I_{\Xi} \otimes \nabla_X + \nabla^{\Xi}_X \otimes I_V .$$

This preserves the YM functional, and even the supersymmetric YM functional, BPS states etc., relevant to string theory. I.e. the 2 projective modules give the same "physics"; a version of "duality". Later Schwarz papers do more on this.

Schwarz also shows:

<u>Theorem</u>. If $T^\infty_{\theta_1}$ and $T^\infty_{\theta_2}$ are gauge Morita equivalent, then θ_1 and θ_2 are in the same $SO(d, d; \mathbb{Z})$ orbit.

2003 Hanfeng Li: Showed that if $g\theta$ is defined (i.e. $(c\theta + d)^{-1}$ exists) for a $g \in SO(d, d; \mathbb{Z})$, then $T_\theta \overset{M}{\simeq} T_{g\theta}$, while T_θ^∞ and $T_{g\theta}^\infty$ are gauge Morita equivalent. (i.e. he removes the "generic" restriction.)

Thus, $T_{\theta_1}^\infty$ and $T_{\theta_2}^\infty$ are gauge Morita equivalent iff θ_1 and θ_2 are in the same $SO(d, d; \mathbb{Z})$ orbit.

2003 Xiang Tang and A. Weinstein:

Give a geometric framework for why T_θ and $T_{g\theta}$ are $\overset{M}{\simeq}$.

Uses a generalization of Poisson structures called "Dirac structures". $SO(d,d;\mathbb{Z})$ has a genuine action on the constant Dirac structures on T^d.

To each Dirac structure is associated a collection of $\overset{M}{\simeq}$ C^*-algebras. And the algebras for Dirac structures in a given $SO(d,d;\mathbb{Z})$-orbit are all $\overset{M}{\simeq}$.

2003 G. Elliott and Hanfeng Li:

Theorem: Let J_d be the set of θ's satisfying a certain Diophantine condition ("generic"). If $T^\infty_{\theta_1} \cong T^\infty_{\theta_2}$ and $\theta_1 \in J_d$, then $\theta_2 \in J_d$, and, θ_1 and θ_2 are in the same $SO(d,d;\mathbb{Z})$-orbit.

So what remains for smooth case is to remove "generic" above.

Case of T_θ still open.