Kac Moody Lie algebras
Quantum groups
Superalgebras

J. Bernstein

Zhang $osp(1|2n)$

$U_q(osp(1|2n)) \simeq U_{-q}(O(2n+1))$

Lanzmann

Hopf algebras

K. M. Lie algebras

$\mathfrak{h}$ - abelian Lie algebra

$\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ simple roots

$X_1, \dots, X_n$
$Y_1, \dots, Y_n$ Chevalley generators

(1) $[h, X_i] = \alpha_i(h) X_i$
$[h, Y_i] = -\alpha_i(h) Y_i$

(2) $[X_i, Y_j] = \delta_{ij} h_i \in \mathfrak{h}$

$\tilde{\mathfrak{g}}$ Lie algebra with (1) & (2)

I the max. ideal, s.t. $\mathfrak{h} \cap I = \{0\}$ $\mathfrak{g} = \tilde{\mathfrak{g}} / I$

Remark. (2) $\Leftrightarrow$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

$\mathfrak{n}^+$ gen. by X_i

$\mathfrak{n}^-$ — " — Y_i

$\alpha_i(h_j) = a_{ij}$

$a_{ii} = 2$

$a_{ij} \in \mathbb{Z}_-$ $i \neq j$

I is determ. by Serre's relations.

Superalgebras

$p(X_i) = p(Y_i) \in \mathbb{Z}_2$

Quantum group: D-J univ. enveloping $K_i = e^{\hbar h_i}$

L free abelian group, fin. generated

$\mathcal{F}(L)$ ring of all functions on L

$\Delta : \mathcal{F}(L) \to \mathcal{F}(L \times L)$

$\Delta f(\alpha, \beta) = f(\alpha + \beta)$

Notation

$$f^{\alpha}(t) \overset{\text{def}}{=} f(t + \alpha)$$

A Hopf alg. gen. by $\mathcal{F}(L)$, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$

$\alpha_1, \ldots, \alpha_n \in L$ lin. indep.

1) $f X_i = X_i f^{\alpha_i}$ (1)

$f Y_i = Y_i f^{-\alpha_i}$

(2) A^+ is a subalg. gen. by X_i, and $\mathcal{F}(L)$

A^- — " — by Y_i and $\mathcal{F}(L)$

$$A = A^- A^+$$

$$\Delta : A \to A \hat{\otimes} A$$

$$A \hat{\otimes} A = \mathcal{F}(L \times L) \otimes_{\mathcal{F}(L) \otimes \mathcal{F}(L)} A \otimes A$$

$$\Delta A^{\pm} \subset A^{\pm} \hat{\otimes} A^{\pm}$$

Thm. Any Hopf algebra, satisfying (1) & (2) is isomorph

$[X_i, Y_j] = \delta_{ij} \gamma_i$

$\Delta X_i = X_i \otimes 1 + \varphi_i \otimes X_i$

$\Delta Y_i = Y_i \otimes \psi_i + 1 \otimes Y_i$

$\varphi_i, \psi_i : L \longrightarrow \mathbb{C}^*$

$\varphi_i(\alpha_j) = \psi_j(-\alpha_i)$ $\quad (*)$

If $\varphi_i \neq \psi_i$ $\quad \gamma_i = \varphi_i - \psi_i$

If $\varphi_i = \psi_i$ $\quad \gamma_i = \varphi_i h_i$

$h_i : L \rightarrow \mathbb{C}$ additive character

Example $n = 1$

$L = \mathbb{Z}\alpha$ $\qquad q = \varphi(\alpha)$

(1) $q = 1$ $\qquad \psi = \varphi^{-1}$ (*)

$[X, Y] = h$ $\qquad h \neq 0$

$\mathcal{U}(\mathfrak{sl}_2)$

(2) $q = -1$; $\varepsilon : L \longrightarrow \mathbb{C}^*$

$\varepsilon(\alpha) = -1$

$\tilde{Y} = \varepsilon Y$

$X\tilde{Y} + \tilde{Y}X = h$ $\qquad h \neq 0$

$\Delta X = X \otimes 1 + \varepsilon \otimes X$

$\Delta \tilde{Y} = \tilde{Y} \otimes 1 + \varepsilon \otimes \tilde{Y}$

$osp(1|2)$ $\qquad$ sign rule

ε defines $\mathbb{Z}_2$-grading

$q \neq \pm 1$

$q \neq q^{-1}$

$\mathcal{U}_q(sl_2)$ q not root of unity

$[X, Y] = q - q^{-1}$

$\mathcal{U}_{-q}(osp(1|2))$

$\tilde{Y} = \varepsilon Y$

$\mathcal{U}_{-q}(osp(1|2)) \subset A \supset \mathcal{U}_q(sl_2)$

$A = F(L)\,\mathcal{U}_q(sl_2) = F(L)\,\mathcal{U}_{-q}(\hat{osp}(1|2))$

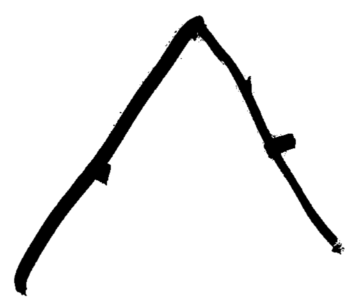
Representations over A

$$M = \bigoplus_{\mu \in L} M_\mu$$

$M_\mu = \{ m \in M \mid fm = f(\mu) m$

$\mathcal{O}$ category, weights are bounded from above

Monoidal category



$M \otimes N$

Braded category

$M \otimes N \simeq N \otimes M$

$\mu \leq \lambda$

$\lambda - \mu = \sum n_i \alpha_i$

$n_i \in \mathbb{Z}_{\geq 0}$

Twisting A, A' the same multiplication

$A \sim A'$

$\Delta' = B \Delta B^{-1}$ $\quad B : L \times L \to \mathbb{C}^*$ bicharacter

$A \; A'$

$$\frac{\varphi_i}{\psi_i} = \frac{\varphi_i'}{\psi_i'}$$

$$q_i = \varphi_i(\alpha_i) = \varphi_i'(\alpha_i)$$

Equivalence classes are parametrized by $\chi_1, \dots, \chi_n : L \to \mathbb{C}^*$

$q_1, \dots, q_n \in \mathbb{C}^*$

$(*)$ $\quad \chi_i(\alpha_j) = \chi_j(\alpha_i)$

About Serre's relation

J the maximal Hopf. ideal, $J \cap F(L)$

$$\mathcal{U} = A / J$$

Singular case

$$q_i = \varphi_i \quad i = 1, \dots, n$$

$$(*) \qquad q_i^2 = 1$$

<u>Theorem</u>. In a singular case any Hopf algebra ((1) and (2)) is equivalent to an extensions of some K. M. Lie superalgebra

$$\mathcal{U}(\mathfrak{g}) \subset \mathcal{U}$$

$$\mathcal{U} = \mathcal{F}(L)\,\mathcal{U}(\mathfrak{g})$$

$q_i = \pm 1$ define the parity on X_i, Y_i

$$\Delta(X_i) = X_i \otimes 1 + \varepsilon \otimes X_i$$

$M \otimes N \qquad X_i(m \otimes n) = X_i m \otimes n + (-1)^{p(X_i)p(m)} m \otimes X_i n$

Assume that
$q_1, \dots, q_n$ are not
roots of unity

- $\mathcal{O} \supset \mathcal{F}$ integrable modules
$\mathcal{M}$ is integrable if
$Y_1, \dots, Y_n$ act locally
nilpotently on M
$\mathcal{F}$ is a braded subcategory

$V(\mu)$ - highest weight
irred. modules

$P = \{\mu \in L \mid V(\mu) \in \mathcal{F}\}$

- $P \ni \mu, \nu \mid \mu + \nu \in P$

P is *large* if for any i
there exist $\lambda \in P$, $\chi_i(\lambda) \neq 1$

Theorem. $q_1, \dots, q_n$ are not roots of unity

P is large

then $\chi_i(\alpha_j) = q_i^{a_{ij}}$

for $a_{ij} \in \mathbb{Z}_{\leq 0}$. $i \neq j$

$\chi_i(\alpha_i) = a_{ii} = 2$

(a_{ij}) symmetrizable $q_1, \dots, q_n$

$$q_i^{a_{ij}} = q_j^{a_{ji}}$$

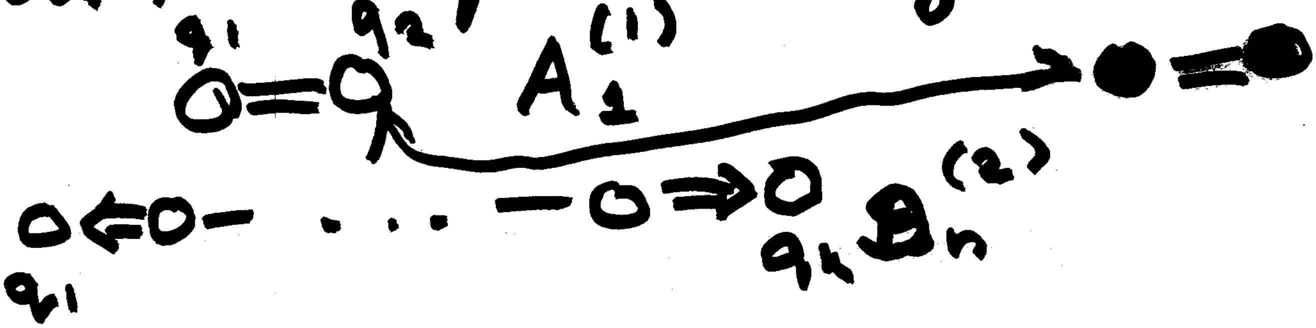
Thm. If any irred. integrable module is f. dim. then

$\mathcal{U} \supset U_q(\mathfrak{g})$

$\mathcal{U} = F(L)\, U_q(\mathfrak{g})$

(a_{ij}) affine

Affine quantum groups



This two diagrams have some new Hopf algebras

$q_2^2 = q_1^2$

$q_2 = q_1$, $q_2 = -q_1$

Kac-Moody Lie superalg.

s. alg. s. that $u(g)$ does not have nilpotents

$osp(1|2n)$

$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ three diff.

"quantum groups

$q_1^3 = q_2^3$

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q_i are roots of unity

$\mathcal{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is a s.alg. with a nil

$\mathcal{U}_q(\mathfrak{sl}(1|2))$

$\mathcal{U}_q(\mathfrak{sl}(2))$ $q^n = 1$.