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CONSTRAINTS & SUPERSPIN FOR SUPERPOINCARÉ ALGEBRAS IN DIVERSE DIMENSIONS

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POINCARÉ LIE ALGEBRA
(PSEUDO-EUCLIDEAN IN D-DIMENS.)

$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots 1)$ $\quad SO(1, D-1)$

$[iJ_{\mu\nu}, P_\rho] = \eta_{\rho\nu} P_\mu - (\mu, \nu)$

$[iJ_{\mu\nu}, J_{\rho\sigma}] = \left(\eta_{\rho\nu} J_{\mu\sigma} - (\mu, \nu)\right) - (\rho, \sigma)$

$[P_\mu, P_\nu] = 0$

REPRESENTATIONS

MASSIVE CASE $\quad P_\mu P^\mu \neq 0$

REST FRAME $\quad P^\mu = m, 0, 0, \cdots 0$

LITTLE GROUP $\quad SO(D-1)$

$J_{ij} = -J_{ji} \quad i, j = 1, \cdots D-1$

$\mu, \nu = 0, 1, \cdots D-1$

MASSLESS CASE $P_\mu P^\mu = 0$

LIGHT-CONE FRAME

$$P^\mu = E, E, 0, \cdots, 0$$
$$\mu = 0 \quad 1 \quad 2 \qquad D-1$$

LITTLE GROUP $ISO(D-2)$

GENERATORS $J_{ij} = -J_{ji}$

$i, j = 2, \cdots D-1$

(ROTATIONS)

$$A_i = J_{i0} - J_{i1}$$

(TRANSATION)

EUCLIDEAN ALGEBRA (NONCOMPACT)

PHYSICAL REQUIREMENT!

$A_i = 0$ (SUSY)

SUPERPOINCARÉ ALGEBRA

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \quad \text{(DIRAC)}$$

Q_α SPINOR FERMIONIC GENERATORS

$$[iJ_{\mu\nu}, Q] = -\frac{1}{2}\Gamma_{\mu\nu} Q$$

$$\Gamma_{\mu\nu} = \frac{i}{2}[\Gamma_\mu, \Gamma_\nu]$$

$$[P_\mu, Q] = 0$$

$$\{Q, \bar{Q}\} = -i2\not{P}$$

$$\bar{Q} = iQ^\dagger\Gamma^0 \qquad \bar{Q} = Q^{*T}$$

$$\not{P} = \Gamma_\mu P^\mu$$

$$\delta\mathcal{O} = \begin{cases} \{Q, \mathcal{O}\} & \mathcal{O} \text{ fermionic} \\ [Q, \mathcal{O}] & \mathcal{O} \text{ bosonic} \end{cases}$$

PAULI-LUBAŃSKI TENSOR

$$W_{\lambda\mu\nu} = P_{\langle\lambda} J_{\mu\nu\rangle} = \frac{1}{3!} \sum_{\text{Perm.}} \pm P_\lambda J_{\mu\nu}$$

RELATIVISTIC FORMULATION OF SPIN AND HELICITY

CAN BE GENERALIZED TO A SUPERSYMMETRIC OBJECT.

FOR THE MASSLESS CASE:

$$\Delta_{\lambda\mu\nu} = W_{\lambda\mu\nu} - \kappa S_{\lambda\mu\nu}$$

FOR THE MASSIVE CASE:

$$C_{\lambda\mu\nu} = W_{\lambda\mu\nu} - \rho S_{\lambda\mu\nu} \qquad \rho = 2\kappa$$

$$S_{\lambda\mu\nu} = \bar{Q} \Gamma_{\lambda\mu\nu} Q$$

$$\Gamma_{\lambda\mu\nu} = \Gamma_{\langle\lambda} \Gamma_\mu \Gamma_{\nu\rangle}$$

$$\delta W_{\lambda\mu\nu} = -\frac{i}{2} P_{<\lambda} \Gamma_{\mu\nu>} Q$$

$$\delta S_{\lambda\mu\nu} = \delta\bar{Q}\Gamma_{\lambda\mu\nu} Q - \bar{Q}\Gamma_{\lambda\mu\nu}\delta Q$$

$$= \begin{cases} -2i \not{P} \Gamma_{\lambda\mu\nu} Q & Q \text{ not Majorana} \\ -4i \not{P} \Gamma_{\lambda\mu\nu} Q & Q \text{ Majorana} \end{cases}$$

MASSLESS CASE:

$$K = \begin{cases} \frac{1}{24} & Q \text{ not Majorana} \\ \frac{1}{48} & Q \text{ Majorana} \end{cases}$$

$$\delta \Delta_{\lambda\mu\nu} = -\frac{1}{12} \Gamma_{\lambda\mu\nu} \not{P} Q$$

$$\delta \not{P} Q \propto P_\mu P^\mu$$

$$\delta P_\mu P^\mu = 0$$

CONSISTENT CONSTRAINTS

$$\Delta_{\lambda\mu\nu} = 0 \quad \not{P} Q = 0 \quad P_\mu P^\mu = 0$$

WHICH FORM A SUPERALGEBRA

COUNTING THE CONSTRAINTS

$$\Delta^{\lambda\mu\nu} = 0 \qquad \binom{D}{3} \; ?$$

$$\Delta^{\rho\lambda\mu\nu} \equiv p^{<\rho}\Delta^{\lambda\mu\nu>} = 0 \quad \text{without using}$$

$$\Delta^{\sigma\rho\lambda\mu\nu} \equiv p^{<\sigma}\Delta^{\rho\lambda\mu\nu>} = 0 \quad \text{without using}$$

$$\vdots$$

$$\Delta^{\tau_1\tau_2\cdots\tau_D} = 0 \qquad \text{by antisymmetry}$$

e.g. for $D = 11$

$$\binom{11}{3} - \binom{11}{4} + \binom{11}{5} \cdots + \binom{11}{11} = \binom{10}{2}$$

MASSIVE CASE $P_\mu P^\mu \neq 0$

$$\rho = \begin{cases} \frac{1}{12} & Q \text{ not Majorana} \\ \frac{1}{24} & Q \text{ Majorana} \end{cases}$$

$$\delta C_{\lambda\mu\nu} = -\frac{1}{12} [\Gamma_{\lambda\mu\nu}, \not{P}] Q$$

$$[P^\lambda \Gamma_{\lambda\mu\nu}, \not{P}] = 0$$

$$C_{\mu\nu} = P^\lambda C_{\lambda\mu\nu} \qquad \delta C_{\mu\nu} = 0$$

$C = C_{\mu\nu} C^{\mu\nu}$ IS A CASIMIR

(FOR THE MASSLESS CASE $C=0$)

Majorana can be imposed in
$D = 2, 3, 4, 8, 9 \pmod 8$;

in $D = 2 \pmod 8$ it can be imposed on a chiral spinor.

A Dirac spinor has 2^n components in $D = 2n$ or $2n+1$ dims.

$$\Delta_{\lambda\mu\nu} \equiv W_{\lambda\mu\nu} - \kappa S_{\lambda\mu\nu}$$

MEANING OF THE CONSTRAINT

$$\Delta_{\lambda\mu\nu} = 0$$

LIGHT-CONE FRAME $P^\mu = (E, E, 0, \cdots, 0)$

$$SO(1, D-1) \supset ISO(D-2)$$

$a, b, c = 0, 1 \qquad i, j, k = 2, \cdots, D-1$

$P^a = E \qquad P^i = 0$

$W_{abc} = W_{ijk} = 0$

$\varepsilon_{ab} = -\varepsilon_{ba} \qquad \varepsilon_{01} = +1$

$$W_{abi} = \varepsilon_{ab} \frac{E}{3} \left(J_{i0} - J_{i1} \right) = \varepsilon_{ab} \frac{E}{3} A_i$$

$W_{aij} = \pm \frac{E}{3} J_{ij} \qquad +$ for $a=1$, $-$ for $a=0$

$S_{abc} = S_{ijk} = 0$

$S_{abi} = 0$

$$S_{aij} = \mp i \, Q^\dagger \Gamma_{ij} Q$$

$$\Delta_{abc} = \Delta_{ijk} = 0$$

$$\Delta_{abi} = \varepsilon_{ab} \frac{E}{3} A_i$$

$$\Delta_{aij} = \pm \frac{E}{3} \left(J_{ij} + 3i\kappa \frac{Q^\dagger \Gamma_{ij} Q}{E} \right)$$

$$\Delta = 0 \quad \longrightarrow \quad A_i = 0$$

$$J_{ij} = -3i\kappa \frac{Q^\dagger \Gamma_{ij} Q}{E}$$

$$\{ Q, Q^\dagger \} = 4E \, \Pi_+$$

$$\not{P} Q = 0 \longrightarrow \quad \Pi_+ Q = Q \ , \ \Pi_- Q = 0$$

$$\Pi_\pm = \frac{1 \pm \Gamma^1 \Gamma^0}{2}$$

Rescale Q to a which will satisfy

$$\{ a, a^\dagger \} = 1 \qquad \left(\tfrac{1}{2}, 0, 0, -\tfrac{1}{2} \right)$$

Ex: in 5D $a = \begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix}$ $\quad J_{ij} = \frac{1}{2} \left(a_1^* a_1 - a_2^* a_2 \right)$

Examples:

$D=10$ $\quad Q$ Majorana - Weyl

4 independent Fermionic oscillators $a_1, \dots, a_4$

$$J_{ij} = \frac{1}{2} \sum_{n=1}^{4} a_n^* a_n - 1$$

eigenvalues for J_{ij} are $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$
with multipl. $(1, 4, 6, 4, 1)$

gauge supermultiplet

$D=11$ $\quad Q$ Majorana

$$J_{ij} = \frac{1}{2} \sum_{n=1}^{8} a_n^* a_n - 2$$

eigenvalues are $(\pm 2, \pm \frac{3}{2}, \pm 1, \pm \frac{1}{2}, 0)$
with multipl. $(1, 8, 28, 56, 70)$

128 Fermionic

128 Bosonic $\quad$ eleven-dim supergravity

MASSIVE CASE

REST FRAME $p^\mu = (m, 0, \ldots, 0)$

LITTLE GROUP $SO(D-1)$

$\not{p} = m\Gamma_0 \qquad \{Q, Q^\dagger\} = 2m$

$a = \dfrac{Q}{\sqrt{2m}} \qquad \{a, a^\dagger\} = 1$

$$[iJ_{ij}, a] = -\frac{1}{2}\Gamma_{ij}\, a$$

$$[iJ_{ij}, a^\dagger] = \frac{1}{2} a^\dagger \Gamma_{ij}$$

$C_{0\mu} = 0 \qquad C_{ij} = -\dfrac{m^2}{3}\left[J_{ij} + 6i\rho\, a^\dagger \Gamma_{ij} a\right]$

DEFINE $T_{ij} = -6i\rho\, a^\dagger \Gamma_{ij} a$

$Y_{ij} = -\dfrac{3}{m^2} C_{ij} \qquad Y_{ij} = J_{ij} - T_{ij}$

$J_{ij} = Y_{ij} + T_{ij}$ $\quad$ Y_{ij} & T_{ij} COMMUTING

THE TOTAL J_{ij} IS THE SUM OF

SUMMARY, in D dims

MASSLESS CASE: THREE CONSTRAINTS WICH FORM A SUPERALGEBRA

$$\Delta_{\lambda\mu\nu} \equiv W_{\lambda\mu\nu} - \kappa \bar{Q} \Gamma_{\lambda\mu\nu} Q = 0$$

$$\not{P} Q = 0 \qquad P^2 = 0$$

$$\delta \Delta_{\lambda\mu\nu} \propto \not{P} Q \qquad \delta \not{P} Q \propto P^2 \qquad \delta P^2 = 0$$

MASSIVE CASE; NO CONSTRAINTS

$$C_{\lambda\mu\nu} = W_{\lambda\mu\nu} - \rho \bar{Q} \Gamma_{\lambda\mu\nu} Q \qquad \rho = 2\kappa$$

$$C_{\mu\nu} = P^\lambda C_{\lambda\mu\nu} \qquad C = C_{\mu\nu} C^{\mu\nu} \text{ Casimir}$$

In the rest frame $C_{ij} \propto Y_{ij}$ superspin angular momentum

RELATION BETWEEN MASSLESS IN D dims AND MASSIVE IN $(D-1)$ dims

$$P_\mu \quad \mu = 0, 1, \cdots, D-1 \qquad P_\mu P^\mu = 0$$

$$P_a = p_a \quad a = 0, 1, \cdots D-2 \qquad P_{D-1} \neq 0$$

$$p_a p^a = -P_{D-1}^2 = -m^2 \neq 0$$

SPINORS $Q = \begin{pmatrix} q \\ \tilde{q} \end{pmatrix}$, $\not{P} Q = 0$ GIVES $\tilde{q}$ IN TERMS OF q AND p

FINALLY, RESTRICT $\Delta_{\lambda\mu\nu}$ TO

Δ_{abc} IN $(D-1)$ dims

ONE FINDS $\Delta_{abc} \neq C_{abc}$ BUT

$p^a \Delta_{abc} = p^a C_{abc} = C_{bc} = 0$ SINCE

IT WAS $\Delta_{\lambda\mu\nu} = 0$

$\longrightarrow$ MASSIVE CASE IN $(D-1)$ dims

WITH $Y \propto C = 0$