

# Maximal lattice-free convex sets in linear subspaces

Amitabh Basu

Carnegie Mellon University  
email: abasu1@andrew.cmu.edu

Michele Conforti

Università di Padova  
email: conforti@math.unipd.it

G erard Cornu ejols

Carnegie Mellon University  
email: gc0v@andrew.cmu.edu

Giacomo Zambelli

Università di Padova  
email: giacomo@math.unipd.it

We consider a model that arises in integer programming, and show that all irredundant inequalities are obtained from maximal lattice-free convex sets in an affine subspace. We also show that these sets are polyhedra. The latter result extends a theorem of Lov asz characterizing maximal lattice-free convex sets in  $\mathbb{R}^n$ .

*Key words:* geometry of numbers; integer programming; maximal lattice-free convex sets; minimal valid inequalities

*MSC2000 Subject Classification:* Primary: 90C11, 90C10; Secondary: 52A20, 52B12

*OR/MS subject classification:* Primary: programming/integer ; Secondary: mathematics/convexity

---

**1. Introduction** The study of maximal lattice-free convex sets dates back to Minkowski’s work on the geometry of numbers. Connections between integer programming and the geometry of numbers were investigated in the 1980s starting with the work of Lenstra [22]. See Lov asz [23] for a survey. Recent work in cutting plane theory [1],[2],[3],[4],[5],[8],[10],[13],[14],[15],[17],[19],[25] has generated renewed interest in the study of maximal lattice-free convex sets. In this paper we further pursue this line of research. In the first part of the paper we consider convex sets in an affine subspace of  $\mathbb{R}^n$  that are maximal with the property of not containing integral points in their relative interior. When this affine subspace is rational, these convex sets are characterized by a result of Lov asz [23]. The extension to irrational subspaces appears to be new, and it has already found an application in the proof of a key result in [10]. It is also used to prove the main result in the second part of this paper: We consider a model that arises in integer programming, and show that all irredundant inequalities are obtained from maximal lattice-free convex sets in an affine subspace.

Let  $W$  be an affine subspace of  $\mathbb{R}^n$ . Assume that  $W$  contains an integral point, i.e.  $W \cap \mathbb{Z}^n \neq \emptyset$ . We say that a set  $B \subset \mathbb{R}^n$  is a *maximal lattice-free convex set in  $W$*  if  $B \subset W$ ,  $B$  is convex,  $B$  has no integral point in its interior with respect to the topology induced on  $W$  by  $\mathbb{R}^n$ , and  $B$  is inclusionwise maximal with these three properties. This definition implies that either  $B$  contains no integral point in its relative interior or  $B$  has dimension strictly less than  $W$ .

The subspace  $W$  is said to be *rational* if it is generated by the integral points in  $W$ . So, if we denote by  $V$  the affine hull of the integral points in  $W$ ,  $V = W$  if and only if  $W$  is rational. If  $W$  is not rational, then the inclusion  $V \subset W$  is strict. When  $W$  is not rational, we will also say that  $W$  is *irrational*. An example of an irrational affine subspace  $W \subseteq \mathbb{R}^3$  is the set of points satisfying the equation  $x_1 + x_2 + \sqrt{2}x_3 = 1$ . The affine hull  $V$  of  $W \cap \mathbb{Z}^3$  is the set of points satisfying the equations  $x_1 + x_2 = 1$ ,  $x_3 = 0$ .

**THEOREM 1.1** *Let  $W \subset \mathbb{R}^n$  be an affine space containing an integral point and  $V$  the affine hull of  $W \cap \mathbb{Z}^n$ . A set  $S \subset W$  is a maximal lattice-free convex set of  $W$  if and only if one of the following holds:*

- (i)  *$S$  is a polyhedron in  $W$  whose dimension equals  $\dim(W)$ ,  $S \cap V$  is a maximal lattice-free convex set of  $V$  whose dimension equals  $\dim(V)$ , the facets of  $S$  and  $S \cap V$  are in one-to-one correspondence and for every facet  $F$  of  $S$ ,  $F \cap V$  is the facet of  $S \cap V$  corresponding to  $F$ ;*

- (ii)  $S$  is an hyperplane of  $W$  of the form  $v + L$ , where  $v \in S$  and  $L \cap V$  is an irrational hyperplane of  $V$ ;
- (iii)  $S$  is a half-space of  $W$  that contains  $V$  on its boundary.

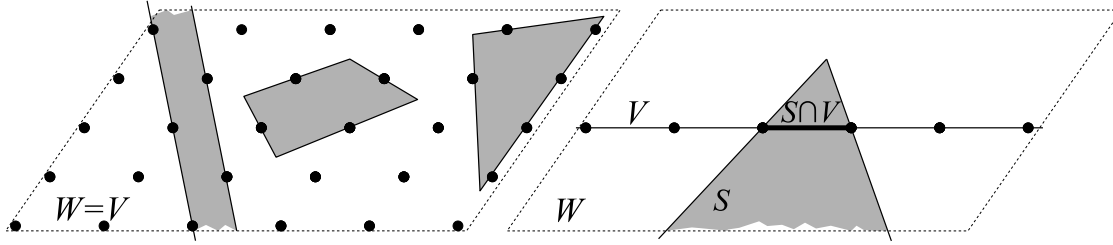


Figure 1: Maximal lattice-free convex sets in a 2-dimensional subspace (Theorem 1.1(i)).

A characterization of maximal lattice-free convex sets of  $V$ , needed in (i) of the previous theorem, is given by the following.

**THEOREM 1.2** (Lovász [23]) *Let  $V$  be a rational affine subspace of  $\mathbb{R}^n$  containing an integral point. A set  $S \subset V$  is a maximal lattice-free convex set of  $V$  if and only if one of the following holds:*

- (i)  $S$  is a polyhedron of the form  $S = P + L$  where  $P$  is a polytope,  $L$  is a rational linear space,  $\dim(S) = \dim(P) + \dim(L) = \dim(V)$ ,  $S$  does not contain any integral point in its relative interior and there is an integral point in the relative interior of each facet of  $S$ ;
- (ii)  $S$  is an affine hyperplane of  $V$  of the form  $v + L$ , where  $v \in S$  and  $L$  is an irrational hyperplane of  $V$ ;

The polyhedron  $S = P + L$  in Theorem 1.2(i) is called a *cylinder over the polytope  $P$*  and can be shown to have at most  $2^{\dim(P)}$  facets [16].

Theorem 1.1 is new and it is used in the proof of our main result about integer programming, Theorem 1.3 below. It is also used to prove the last theorem in [10]. Theorem 1.2 is due to Lovász ([23] Proposition 3.1). Lovász only gives a sketch of the proof and it is not clear how case (ii) in Theorem 1.2 arises in his sketch or in the statement of his proposition. Therefore in Section 2 we will prove both theorems.

Figure 1 shows examples of maximal lattice-free convex sets in a 2-dimensional affine subspace  $W$  of  $\mathbb{R}^3$ . We denote by  $V$  the affine space generated by  $W \cap \mathbb{Z}^3$ . In the first picture  $W$  is rational, so  $V = W$ , while in the second one  $V$  is a subspace of  $W$  of dimension 1.

We now give an example of Theorem 1.1(ii). Let  $W \subseteq \mathbb{R}^4$  be the set of points satisfying the equation  $x_1 + x_2 + x_3 + \sqrt{2}x_4 = 1$ . The affine hull  $V$  of  $W \cap \mathbb{Z}^4$  is the set of points satisfying the equations  $x_1 + x_2 + x_3 = 1$ ,  $x_4 = 0$ . The set  $S \subset W$  defined by the equations  $x_1 + x_2 + x_3 + \sqrt{2}x_4 = 1$ ,  $x_1 + \sqrt{2}x_2 = 1$  satisfies Theorem 1.1(ii). Indeed,  $\dim(W) = \dim(S) + 1 = 3$ . Furthermore,  $\dim(V) = 2$  and  $S \cap V$  is the line satisfying the equations  $x_1 + x_2 + x_3 = 1$ ,  $x_1 + \sqrt{2}x_2 = 1$ ,  $x_4 = 0$  and it is an irrational subspace since the only integral point it contains is  $(1, 0, 0, 0)$ .

Next we highlight the relation between lattice-free convex sets and valid inequalities in integer programming. This was first observed by Balas [6].

Suppose we consider  $q$  rows of the optimal tableau of the LP relaxation of a given MILP, relative to  $q$  basic integer variables  $x_1, \dots, x_q$ . Let  $s_1, \dots, s_k$  be the nonbasic variables, and  $f \in \mathbb{R}^q$  be the vector of components of the optimal basic feasible solution. The tableau restricted to these  $q$  rows is of the form

$$x = f + \sum_{j=1}^k r^j s_j, \quad x \geq 0 \text{ integral}, \quad s \geq 0, \quad \text{and } s_j \in \mathbb{Z}, j \in I,$$

where  $r^j \in \mathbb{R}^q$ ,  $j = 1, \dots, k$ , and  $I$  denotes the set of integer nonbasic variables. Gomory [18] proposed to consider the relaxation of the above problem obtained by dropping the nonnegativity conditions  $x \geq 0$ . This gives rise to the so called *corner polyhedron*. A further relaxation is obtained by also dropping the integrality conditions on the nonbasic variables, obtaining the mixed-integer set

$$x = f + \sum_{j=1}^k r^j s_j, \quad x \in \mathbb{Z}^q, \quad s \geq 0.$$

Note that, since  $x \in \mathbb{R}^q$  is completely determined by  $s \in \mathbb{R}^k$ , the above is equivalent to

$$f + \sum_{j=1}^k r^j s_j \in \mathbb{Z}^q, \quad s \geq 0. \tag{1}$$

We denote by  $R_f(r^1, \dots, r^k)$  the set of points  $s$  satisfying (1). The above relaxation was studied by Andersen et al. [1] in the case of two rows and Borozan and Cornuéjols [10] for the general case. In these papers they showed that the irredundant valid inequalities for  $R_f(r^1, \dots, r^k)$  correspond to maximal lattice-free convex sets in  $\mathbb{R}^q$ . In [1, 10] data are assumed to be rational. Here we consider the case where  $f, r^1, \dots, r^k$  may have irrational entries.

Let  $W = \langle r^1, \dots, r^k \rangle$  be the linear space generated by  $r^1, \dots, r^k$ . Note that, for every  $s \in R_f(r^1, \dots, r^k)$ , the point  $f + \sum_{j=1}^k r^j s_j \in (f + W) \cap \mathbb{Z}^q$ , hence we assume  $f + W$  contains an integral point. Let  $V$  be the affine hull of  $(f + W) \cap \mathbb{Z}^q$ . Notice that  $f + W$  and  $V$  coincide if and only if  $W$  is a rational space. Borozan and Cornuéjols [10] proposed to study the following *semi-infinite relaxation*, which is a special case of Gomory and Johnson’s group problem [20]. Let  $R_f(W)$  be the set of points  $s = (s_r)_{r \in W}$  of  $\mathbb{R}^W$  satisfying

$$\begin{aligned} f + \sum_{r \in W} r s_r &\in \mathbb{Z}^q \\ s_r &\geq 0, \quad r \in W \\ s &\in \mathcal{W} \end{aligned} \tag{2}$$

where  $\mathcal{W}$  is the set of all  $s \in \mathbb{R}^W$  with *finite support*, i.e. the set  $\{r \in W \mid s_r > 0\}$  has finite cardinality. Notice that  $R_f(r^1, \dots, r^k) = R_f(W) \cap \{s \in \mathcal{W} \mid s_r = 0 \text{ for all } r \neq r^1, \dots, r^k\}$ .

Given a function  $\psi : W \rightarrow \mathbb{R}$  and  $\alpha \in \mathbb{R}$ , the linear inequality

$$\sum_{r \in W} \psi(r) s_r \geq \alpha \tag{3}$$

is *valid for  $R_f(W)$*  if it is satisfied by every  $s \in R_f(W)$ .

Note that, given a valid inequality (3) for  $R_f(W)$ , the inequality

$$\sum_{j=1}^k \psi(r^j) s_j \geq \alpha$$

is valid for  $R_f(r^1, \dots, r^k)$ . Hence a characterization of valid linear inequalities for  $R_f(W)$  provides a characterization of valid linear inequalities for  $R_f(r^1, \dots, r^k)$ .

Next we observe how maximal lattice-free convex sets in  $f + W$  give valid linear inequalities for  $R_f(W)$ . Let  $B$  be a maximal lattice-free convex set in  $f + W$  containing  $f$  in its interior. Since, by Theorem 1.1,  $B$  is a polyhedron and since  $f$  is in its interior, there exist  $a_1, \dots, a_t \in \mathbb{R}^q$  such that  $B = \{x \in f + W \mid a_i(x - f) \leq 1, i = 1 \dots, t\}$ . We define the function  $\psi_B : W \rightarrow \mathbb{R}$  by

$$\psi_B(r) = \max_{i=1, \dots, t} a_i r. \tag{4}$$

Note that the function  $\psi_B$  is *subadditive*, i.e.  $\psi_B(r) + \psi_B(r') \geq \psi_B(r + r')$ , and *positively homogeneous*, i.e.  $\psi_B(\lambda r) = \lambda \psi_B(r)$  for every  $\lambda \geq 0$ . We claim that

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

is valid for  $R_f(W)$ .

Indeed, let  $s \in R_f(W)$ , and  $x = f + \sum_{r \in W} r s_r$ . Since  $x \in \mathbb{Z}^n$  and  $B$  is lattice-free,  $x \notin \mathbf{int}(B)$ . Then

$$\sum_{r \in W} \psi_B(r) s_r = \sum_{r \in W} \psi_B(r s_r) \geq \psi_B\left(\sum_{r \in W} r s_r\right) = \psi_B(x - f) \geq 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of  $\psi_B$  and the last one follows from the fact that  $x \notin \mathbf{int}(B)$ .

We will show that all nontrivial irredundant valid linear inequalities for  $R_f(W)$  are indeed of the type described above. Furthermore, if  $W$  is irrational, we will see that  $R_f(W)$  is contained in a proper affine subspace of  $\mathcal{W}$ , so each inequality has infinitely many equivalent forms. Note that, by definition of  $\psi_B$ ,  $\psi_B(r) > 0$  if  $r$  is not in the recession cone of  $B$ ,  $\psi_B(r) < 0$  when  $r$  is in the interior of the recession cone of  $B$ , while  $\psi_B(r) = 0$  when  $r$  is on the boundary of the recession cone of  $B$ . We will show that one can always choose a form of the inequality so that  $\psi_B$  is a nonnegative function. We make this more precise in the next theorem.

Given a point  $s \in R_f(W)$ , then  $f + \sum_{r \in W} r s_r \in \mathbb{Z}^q \cap (f + W)$ . Recall that we denote by  $V$  the affine hull of  $\mathbb{Z}^q \cap (f + W)$ . Thus  $R_f(W)$  is contained in the affine subspace  $\mathcal{V}$  of  $\mathcal{W}$  defined as

$$\mathcal{V} = \left\{ s \in \mathcal{W} \mid f + \sum_{r \in W} r s_r \in V \right\}.$$

Observe that, given  $C \in \mathbb{R}^{\ell \times q}$  and  $d \in \mathbb{R}^\ell$  such that  $V = \{x \in f + W \mid Cx = d\}$ , we have

$$\mathcal{V} = \left\{ s \in \mathcal{W} \mid \sum_{r \in W} (Cr) s_r = d - Cf \right\}. \quad (5)$$

Given two valid inequalities  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  and  $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$  for  $R_f(W)$ , we say that they are *equivalent* if there exist  $\rho > 0$  and  $\lambda \in \mathbb{R}^\ell$  such that  $\psi(r) = \rho \psi'(r) + \lambda^T Cr$  and  $\alpha = \rho \alpha' + \lambda^T (d - Cf)$ . Note that, if two valid inequalities  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  and  $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$  for  $R_f(W)$  are equivalent, then  $\mathcal{V} \cap \{s \mid \sum_{r \in W} \psi(r) s_r \geq \alpha\} = \mathcal{V} \cap \{s \mid \sum_{r \in W} \psi'(r) s_r \geq \alpha'\}$ .

A linear inequality  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  that is satisfied by every element in  $\{s \in \mathcal{V} \mid s_r \geq 0 \text{ for every } r \in W\}$  is said to be *trivial*.

We say that inequality  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  *dominates* inequality  $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$  if  $\psi(r) \leq \psi'(r)$  for all  $r \in W$ . Note that, for any  $\bar{s} \in \mathcal{W}$  such that  $\bar{s}_r \geq 0$  for all  $r \in W$ , if  $\bar{s}$  satisfies the first inequality, then  $\bar{s}$  also satisfies the second. A valid inequality  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  for  $R_f(W)$  is *minimal* if it is not dominated by any valid linear inequality  $\sum_{r \in W} \psi'(r) s_r \geq \alpha'$  for  $R_f(W)$  such that  $\psi' \neq \psi$ . It is not obvious that nontrivial valid linear inequalities are dominated by minimal ones. We will show that this is the case. Note that it is not even obvious that minimal valid linear inequalities exist.

We will show that, for any maximal lattice-free convex set  $B$  of  $f + W$  with  $f$  in its interior, the inequality  $\sum_{r \in W} \psi_B(r) s_r \geq 1$  is a minimal valid inequality for  $R_f(W)$ . The main result is a converse, stated in the next theorem. We need the notion of equivalent inequalities, which define the same region in  $\mathcal{V}$ .

**THEOREM 1.3** *Every nontrivial valid linear inequality for  $R_f(W)$  is dominated by a nontrivial minimal valid linear inequality for  $R_f(W)$ .*

*Every nontrivial minimal valid linear inequality for  $R_f(W)$  is equivalent to an inequality of the form*

$$\sum_{r \in W} \psi_B(r) s_r \geq 1$$

*such that  $\psi_B(r) \geq 0$  for all  $r \in W$  and  $B$  is a maximal lattice-free convex set in  $f + W$  with  $f$  in its interior.*

This theorem generalizes earlier results of Borozan and Cornuéjols [10]. In their setting it is immediate that all valid linear inequalities are of the form  $\sum_{r \in W} \psi(r) s_r \geq 1$  with  $\psi$  nonnegative. From this, it follows easily that  $\psi$  must be equal to  $\psi_B$  for some maximal lattice-free convex set  $B$ . The proof is

much more complicated for the case of  $R_f(W)$  when  $W$  is an irrational space. In this case, valid linear inequalities might have negative coefficients. For minimal inequalities, however, Theorem 1.3 shows that there always exists an equivalent one where all coefficients are nonnegative. The function  $\psi_B$  is nonnegative if and only if the recession cone of  $B$  has empty interior. Although there are nontrivial minimal valid linear inequalities arising from maximal lattice-free convex sets whose recession cone is full dimensional, Theorem 1.3 states that there always exists a maximal lattice-free convex set whose recession cone is not full dimensional that gives an equivalent inequality. A crucial ingredient in showing this is a new result about sublinear functions proved in [9].

In light of Theorem 1.3, it is a natural question to ask what is the subset of  $\mathcal{W}$  obtained by intersecting the set of nonnegative elements of  $\mathcal{V}$  with all half-spaces defined by inequalities  $\sum_{r \in W} \psi(r)s_r \geq 1$  as in Theorem 1.3. In a finite dimensional space, the intersection of all half-spaces containing a given convex set  $C$  is the closure of  $C$ . Things are more complicated in infinite dimension. First of all, while in finite dimension all norms are topologically equivalent, and thus the concept of closure does not depend on the choice of a specific norm, in infinite dimension different norms may produce different topologies. Secondly, in finite dimensional spaces linear functions are always continuous, while in infinite dimension there always exist linear functions that are not continuous. In particular, half-spaces (i.e. sets of points satisfying a linear inequality) are not always closed in infinite dimensional spaces (see Conway [12] for example).

To illustrate this, note that if  $\mathcal{W}$  is endowed with the Euclidean norm, then  $\mathbf{0} = (0)_{r \in W}$  belongs to the closure of  $\text{conv}(R_f(W))$  with respect to this norm, as shown next. Let  $\bar{x}$  be an integral point in  $f + W$  and let  $\bar{s}$  be defined by

$$\bar{s}_r = \begin{cases} \frac{1}{k} & \text{if } r = k(\bar{x} - f), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, for every choice of  $k$ ,  $\bar{s} \in R_f(W)$ , and for  $k$  that goes to infinity the point  $\bar{s}$  is arbitrarily close to  $\mathbf{0}$  with respect to the Euclidean

distance. Now, given a valid linear inequality  $\sum_{r \in W} \psi(r)s_r \geq 1$  for  $\text{conv}(R_f(W))$ , since  $\sum_{r \in W} \psi(r)\mathbf{0} = 0$  the hyperplane  $\mathcal{H} = \{s \in \mathcal{W} : \sum_{r \in W} \psi(r)s_r = 1\}$  separates strictly  $\text{conv}(R_f(W))$  from  $\mathbf{0}$  even though  $\mathbf{0}$  is in the closure of  $\text{conv}(R_f(W))$ . This implies that  $\mathcal{H}$  is not a closed hyperplane of  $\mathcal{W}$ , and in particular the function  $s \mapsto \sum_{r \in W} \psi(r)s_r$  is not continuous with respect to the Euclidean norm on  $\mathcal{W}$ .

A nice answer to our question is given by considering a different norm on  $\mathcal{W}$ . We endow  $\mathcal{W}$  with the norm  $\|\cdot\|_H$  defined by

$$\|s\|_H = |s_0| + \sum_{r \in W \setminus \{0\}} \|r\| |s_r|.$$

It is straightforward to show that  $\|\cdot\|_H$  is indeed a norm. Given  $A \subset \mathcal{W}$ , we denote by  $\bar{A}$  the closure of  $A$  with respect to the norm  $\|\cdot\|_H$ .

Let  $\mathcal{B}_W$  be the family of all maximal lattice-free convex sets of  $W$  with  $f$  in their interior.

THEOREM 1.4

$$\overline{\text{conv}}(R_f(W)) = \left\{ s \in \mathcal{V} \mid \begin{array}{l} \sum_{r \in W} \psi_B(r)s_r \geq 1 \\ s_r \geq 0 \end{array} \begin{array}{l} B \in \mathcal{B}_W \\ r \in W \end{array} \right\}.$$

Note that Theorems 1.3 and 1.4 are new even when  $W = \mathbb{R}^q$ .

A valid inequality  $\sum_{r \in W} \psi(r)s_r \geq 1$  for  $R_f(W)$  is said to be *extreme* if there do not exist distinct functions  $\psi_1$  and  $\psi_2$  satisfying  $\psi \geq \frac{1}{2}(\psi_1 + \psi_2)$ , such that  $\sum_{r \in W} \psi_i(r)s_r \geq 1$ ,  $i = 1, 2$ , are both valid for  $R_f(W)$ . The above definition is due to Gomory and Johnson [20]. Note that, if an inequality is not extreme, then it is not necessary to define  $\overline{\text{conv}}(R_f(W))$ .

The next theorem exhibits a correspondence between extreme inequalities for the infinite model  $R_f(\mathbb{R}^q)$  and extreme inequalities for some finite problem  $R_f(r^1, \dots, r^k)$  where  $r^1, \dots, r^k \in \mathbb{R}^q$ . The theorem is very similar to a result of Dey and Wolsey [15].

**THEOREM 1.5** *Let  $B$  be a maximal lattice-free convex set in  $\mathbb{R}^q$  with  $f$  in its interior. Let  $L = \text{lin}(B)$  and let  $P = B \cap (f + L^\perp)$ . Then  $B = P + L$ ,  $L$  is a rational space, and  $P$  is a polytope. Let  $v^1, \dots, v^k$  be the vertices of  $P$ , and  $r^{k+1}, \dots, r^{k+h}$  be a rational basis of  $L$ . Define  $r^j = v^j - f$  for  $j = 1, \dots, k$ .*

*Then the inequality  $\sum_{r \in \mathbb{R}^q} \psi_B(r) s_r \geq 1$  is extreme for  $R_f(\mathbb{R}^q)$  if and only if the inequality  $\sum_{j=1}^k s_j \geq 1$  is extreme for  $\text{conv}(R_f(r^1, \dots, r^{k+h}))$ .*

Even though the data in integer programs are typically rational and studying the infinite relaxation (2) for  $W = \mathbb{Q}^q$  seems natural [10, 13], some of its extreme inequalities arise from maximal lattice-free convex sets that are not rational polyhedra [13].

For example, the irrational triangle  $B$  defined by the inequalities  $x_1 + x_2 \leq 2$ ,  $x_2 \geq 1 + \sqrt{2}x_1$ ,  $x_2 \geq 0$  is a maximal lattice-free convex set in the plane, and it gives rise to an extreme valid inequality  $\sum_{r \in \mathbb{Q}^2} \psi_B(r) s_r \geq 1$  for  $R_f(\mathbb{Q}^2)$  for any rational  $f$  in the interior of  $B$ . In fact, every maximal lattice-free triangle gives rise to an extreme valid inequality for  $R_f(\mathbb{Q}^2)$  [13]. Therefore, even when  $W = \mathbb{Q}^q$  in (2), irrational coefficients are needed to describe some of the extreme inequalities for  $R_f(\mathbb{Q}^q)$ . Indeed, it follows from Theorem 1.3 and from [10] that the extreme inequalities for  $R_f(\mathbb{Q}^q)$  are precisely the restrictions to  $\mathbb{Q}^q$  of the extreme inequalities for  $R_f(\mathbb{R}^q)$ . This suggests that the more natural setting for (2) might in fact be  $W = \mathbb{R}^q$ .

The paper is organized as follows. In Section 2 we will state and prove the natural extensions of Theorems 1.1 and 1.2 for general lattices. In Section 3 we prove Theorem 1.3, while in Section 4 we prove Theorem 1.4 and in Section 5 we prove Theorem 1.5.

**2. Maximal lattice-free convex sets** Given  $X \subset \mathbb{R}^n$ , we denote by  $\langle X \rangle$  the linear space generated by the vectors in  $X$ . The underlying field is  $\mathbb{R}$  in this paper. The purpose of this section is to prove Theorems 1.1 and 1.2. For this, we will need to work with general lattices.

**DEFINITION 2.1** *An additive group  $\Lambda$  of  $\mathbb{R}^n$  is said to be finitely generated if there exist vectors  $a_1, \dots, a_m \in \mathbb{R}^n$  such that  $\Lambda = \{\lambda_1 a_1 + \dots + \lambda_m a_m \mid \lambda_1, \dots, \lambda_m \in \mathbb{Z}\}$ .*

*If a finitely generated additive group  $\Lambda$  of  $\mathbb{R}^n$  can be generated by linearly independent vectors  $a_1, \dots, a_m$ , then  $\Lambda$  is called a lattice of the linear space  $\langle a_1, \dots, a_m \rangle$ . The set of vectors  $a_1, \dots, a_m$  is called a basis of the lattice  $\Lambda$ .*

**DEFINITION 2.2** *Let  $\Lambda$  be a lattice of a linear space  $V$  of  $\mathbb{R}^n$ . Given a linear subspace  $L$  of  $V$ , we say that  $L$  is a  $\Lambda$ -subspace of  $V$  if there exists a basis of  $L$  contained in  $\Lambda$ .*

For example, in  $\mathbb{R}^2$ , consider the lattice  $\Lambda$  generated by vectors  $(0, 1)$  and  $(1, 0)$ . The line  $x_2 = 2x_1$  is a  $\Lambda$ -subspace, whereas the line  $x_2 = \sqrt{2}x_1$  is not.

Given  $y \in \mathbb{R}^n$  and  $\varepsilon > 0$ , we will denote by  $B_\varepsilon(y)$  the open ball centered at  $y$  of radius  $\varepsilon$ . Given an affine space  $W$  of  $\mathbb{R}^n$  and a set  $S \subseteq W$ , we denote by  $\mathbf{int}_W(S)$  the interior of  $S$  with respect to the topology induced on  $W$  by  $\mathbb{R}^n$ , namely  $\mathbf{int}_W(S)$  is the set of points  $x \in S$  such that  $B_\varepsilon(x) \cap W \subset S$  for some  $\varepsilon > 0$ . We denote by  $\mathbf{relint}(S)$  the relative interior of  $S$ , that is  $\mathbf{relint}(S) = \mathbf{int}_{\text{aff}(S)}(S)$ .

**DEFINITION 2.3** *Let  $\Lambda$  be a lattice of a linear space  $V$  of  $\mathbb{R}^n$ , and let  $W$  be a linear space of  $\mathbb{R}^n$  containing  $V$ . A set  $S \subset \mathbb{R}^n$  is said to be a  $\Lambda$ -free convex set of  $W$  if  $S \subset W$ ,  $S$  is convex and  $\Lambda \cap \mathbf{int}_W(S) = \emptyset$ , and  $S$  is said to be a maximal  $\Lambda$ -free convex set of  $W$  if it is not properly contained in any  $\Lambda$ -free convex set.*

The next two theorems are restatements of Theorems 1.1 and 1.2 for general lattices.

**THEOREM 2.1** *Let  $\Lambda$  be a lattice of a linear space  $V$  of  $\mathbb{R}^n$ , and let  $W$  be a linear space of  $\mathbb{R}^n$  containing  $V$ . A set  $S \subset \mathbb{R}^n$  is a maximal  $\Lambda$ -free convex set of  $W$  if and only if one of the following holds:*

- (i)  *$S$  is a polyhedron in  $W$ ,  $\dim(S) = \dim(W)$ ,  $S \cap V$  is a maximal  $\Lambda$ -free convex set of  $V$ , the facets of  $S$  and  $S \cap V$  are in one-to-one correspondence and for every facet  $F$  of  $S$ ,  $F \cap V$  is the facet of  $S \cap V$  corresponding to  $F$ ;*

- (ii)  $S$  is an affine hyperplane of  $W$  of the form  $S = v + L$  where  $v \in S$  and  $L \cap V$  is a hyperplane of  $V$  that is not a lattice subspace of  $V$ ;
- (iii)  $S$  is a half-space of  $W$  that contains  $V$  on its boundary.

**THEOREM 2.2** *Let  $\Lambda$  be a lattice of a linear space  $V$  of  $\mathbb{R}^n$ . A set  $S \subset \mathbb{R}^n$  is a maximal  $\Lambda$ -free convex set of  $V$  if and only if one of the following holds:*

- (i)  $S$  is a polyhedron of the form  $S = P + L$  where  $P$  is a polytope,  $L$  is a  $\Lambda$ -subspace of  $V$ ,  $\dim(S) = \dim(P) + \dim(L) = \dim(V)$ ,  $S$  does not contain any point of  $\Lambda$  in its interior and there is a point of  $\Lambda$  in the relative interior of each facet of  $S$ ;
- (ii)  $S$  is an affine hyperplane of  $V$  of the form  $S = v + L$  where  $v \in S$  and  $L$  is not a  $\Lambda$ -subspace of  $V$ .

**2.1 Proof of Theorem 2.1** We assume Theorem 2.2 holds. Its proof will be given in the next section.

( $\Rightarrow$ ) Let  $S$  be a maximal  $\Lambda$ -free convex set of  $W$ . We show that one of (i) – (iii) holds. If  $V = W$ , then (iii) cannot occur and either (i) or (ii) follows from Theorem 2.2. Thus we assume  $V \subset W$ .

Assume first that  $\dim(S) < \dim(W)$ . Then there exists a hyperplane  $H$  of  $W$  containing  $S$ , and since  $\mathbf{int}_W(H) = \emptyset$ , then  $S = H$  by maximality of  $S$ . Since  $S$  is a hyperplane of  $W$ , then either  $V \subseteq S$  or  $S \cap V$  is a hyperplane of  $V$ . If  $V \subseteq S$ , then let  $K$  be one of the two half spaces of  $W$  separated by  $S$ . Then  $\mathbf{int}_W(K) \cap \Lambda = \emptyset$ , contradicting the maximality of  $S$ . Hence  $S \cap V$  is a hyperplane of  $V$ . We show that  $P = S \cap V$  is a maximal  $\Lambda$ -free convex set of  $V$ . Indeed, let  $K$  be a convex set in  $V$  such that  $\mathbf{int}_V(K) \cap \Lambda = \emptyset$  and  $P \subseteq K$ . Since  $\text{conv}(S \cup K) \cap V = K$ , then  $\mathbf{int}_W(\text{conv}(S \cup K) \cap \Lambda) = \emptyset$ . By maximality of  $S$ ,  $S = \text{conv}(S \cup K)$ , hence  $P = K$ .

Given  $v \in P$ ,  $S = v + L$  for some hyperplane  $L$  of  $W$ , and  $P = v + (L \cap V)$ . Applying Theorem 2.2 to  $P$ , we get that  $L \cap V$  is not a lattice subspace of  $V$ , and case (ii) holds.

So we may assume  $\dim(S) = \dim(W)$ . Since  $S$  is convex, then  $\mathbf{int}_W(S) \neq \emptyset$ . We consider two cases.

*Case 1.*  $\mathbf{int}_W(S) \cap V = \emptyset$ .

Since  $\mathbf{int}_W(S)$  and  $V$  are nonempty disjoint convex sets, there exists a hyperplane separating them, i.e. there exist  $\alpha \in \mathbb{R}^n$  and  $\beta \in \mathbb{R}$  such that  $\alpha x \geq \beta$  for every  $x \in S$  and  $\alpha x \leq \beta$  for every  $x \in V$ . Since  $V$  is a linear space, then  $\alpha x = 0$  for every  $x \in V$ , hence  $\beta \geq 0$ . Then the half space  $H = \{x \in W \mid \alpha x \geq 0\}$  contains  $S$  and  $V$  lies on the boundary of  $H$ . Hence  $H$  is a maximal  $\Lambda$ -free convex set of  $W$  containing  $S$ , therefore  $S = H$  by the maximality assumption, so (iii) holds.

*Case 2.*  $\mathbf{int}_W(S) \cap V \neq \emptyset$ .

We claim that

$$\mathbf{int}_W(S) \cap V = \mathbf{int}_V(S \cap V). \tag{6}$$

To prove this claim, notice that the direction  $\mathbf{int}_W(S) \cap V \subseteq \mathbf{int}_V(S \cap V)$  is straightforward. Conversely, let  $x \in \mathbf{int}_V(S \cap V)$ . Then there exists  $\varepsilon > 0$  such that  $B_\varepsilon(x) \cap V \subseteq S$ . Since  $\mathbf{int}_W(S) \cap V \neq \emptyset$ , there exists  $y \in \mathbf{int}_W(S) \cap V$ . Then there exists  $\varepsilon' > 0$  such that  $B_{\varepsilon'}(y) \cap W \subseteq S$ . We may assume  $y \neq x$  since otherwise the result holds. Since  $x \in \mathbf{int}(B_\varepsilon(x))$ , there exists  $z \in \mathbf{int}_W(B_\varepsilon(x))$  such that  $x$  is in the relative interior of the segment  $yz$ . Since  $x, y \in V$ ,  $z \in V$  and therefore  $z \in S$ . The ball  $B_\delta(x)$  with radius  $\delta = \varepsilon' \frac{\|x-z\|}{\|y-z\|}$  is contained in  $\text{conv}(\{z\} \cup B_{\varepsilon'}(y))$ . By convexity of  $S$ ,  $\text{conv}(\{z\} \cup (B_{\varepsilon'}(y) \cap W)) \subseteq S$  and therefore  $(B_\delta(x) \cap W) \subseteq S$ . Thus  $x \in \mathbf{int}_W(S)$ . Since  $x \in V$ , it follows that  $x \in \mathbf{int}_W(S) \cap V$ .

Let  $P = S \cap V$ . By (6) and because  $\mathbf{int}_W(S) \cap \Lambda = \emptyset$ , we have  $\mathbf{int}_V(P) \cap \Lambda = \emptyset$ . We show that  $P$  is a maximal  $\Lambda$ -free convex set of  $V$ . Indeed, let  $K$  be a convex set in  $V$  such that  $\mathbf{int}_V(K) \cap \Lambda = \emptyset$  and  $P \subseteq K$ . Since  $\text{conv}(S \cup K) \cap V = K$ , Claim (6) implies that  $\mathbf{int}_W(\text{conv}(S \cup K)) \cap \Lambda = \emptyset$ . By maximality,  $S = \text{conv}(S \cup K)$ , hence  $P = K$ .

Since  $\dim(P) = \dim(V)$ , by Theorem 2.2 applied to  $P$ ,  $P$  is a polyhedron with a point of  $\Lambda$  in the relative interior of each of its facets. Let  $F_1, \dots, F_t$  be the facets of  $P$ . For  $i = 1, \dots, t$ , let  $z_i$  be a point in  $\mathbf{relint}(F_i) \cap \Lambda$ . By (6),  $z_i \notin \mathbf{int}_W(S)$ . By the separation theorem, there exists a half-space  $H_i$  of  $W$  containing  $\mathbf{int}_W(S)$  such that  $z_i \notin \mathbf{int}_W(H_i)$ . Notice that  $F_i$  is on the boundary of  $H_i$ . Then  $S \subseteq \bigcap_{i=1}^t H_i$ . By construction  $\mathbf{int}_W(\bigcap_{i=1}^t H_i) \cap \Lambda = \emptyset$ , hence by maximality of  $S$ ,  $S = \bigcap_{i=1}^t H_i$ . For every  $j = 1, \dots, t$ ,  $\mathbf{int}_W(\bigcap_{i \neq j} H_i)$  contains  $z_j$ . Therefore  $H_j$  defines a facet of  $S$  for  $j = 1, \dots, t$ .

( $\Leftarrow$ ) Let  $S$  be a set in  $\mathbb{R}^n$  satisfying one of (i), (ii), (iii). Clearly  $S$  is a convex set in  $W$  and  $\mathbf{int}_W(S) \cap \Lambda = \emptyset$ , so we only need to prove maximality. If  $S$  satisfies (iii), then this is immediate. This is also immediate when  $S$  satisfies (i) and  $V = W$ . So we may assume that either  $S$  satisfies (i) and  $\dim(V) < \dim(W)$ , or  $S$  satisfies (ii).

Suppose that there exists a closed convex set  $K \subset W$  strictly containing  $S$  such that  $\mathbf{int}_W(K) \cap \Lambda = \emptyset$ . Let  $w \in K \setminus S$ . Then  $\text{conv}(S \cup \{w\}) \subseteq K$ . To conclude the proof of the theorem, it suffices to prove that  $S \cap V$  is strictly contained in  $\text{conv}(S \cup \{w\}) \cap V$ . Indeed, by maximality of  $S \cap V$ , this claim implies that the set  $\mathbf{int}_V(\text{conv}(S \cup \{w\}) \cap V)$  contains a point in  $\Lambda$ . Now  $\text{conv}(S \cup \{w\}) \subseteq K$  implies that  $\mathbf{int}_W(K)$  contains a point of  $\Lambda$ , a contradiction.

It only remains to prove that  $S \cap V \subset \text{conv}(S \cup \{w\}) \cap V$ . This is clear when  $S$  is a hyperplane satisfying (ii). The statement is also clear if  $w \in V$ . Assume now that  $S$  is a polyhedron satisfying (i) and  $\dim(V) < \dim(W)$  and that  $w \notin V$ . Let  $F$  be a facet of  $S$  that separates  $w$  from  $S$ . If  $F \cap V$  is contained in a proper face of  $F$ , then  $F \cap V$  is contained in at least two facets of  $S$ , a contradiction to the one-to-one correspondence property. So  $F \cap V$  is not contained in proper face of  $F$ . Therefore, there exists  $p \in \mathbf{relint}(F) \cap V$ . Choose  $\varepsilon > 0$  such that  $B_\varepsilon(p) \cap \text{aff}(F) \subseteq F$ . Note that  $F \not\subseteq V$  since otherwise  $F \cap V = F$  but this is a contradiction since  $\dim(S \cap V) = \dim(V) < \dim(W) = \dim(S)$ , and  $\dim(F \cap V) = \dim(S \cap V) - 1$ ,  $\dim(F) = \dim(S) - 1$ .

Let  $W' = \text{aff}(V \cup \{w\})$ . Note that  $V$  and  $\text{aff}(F) \cap W'$  are distinct affine hyperplanes of  $W'$ . Let  $H, H'$  be the two open half-spaces of  $W'$  defined by  $V$ , and assume w.l.o.g. that  $w \in H'$ . Since  $p \in V$ ,  $H \cap \text{aff}(F) \cap B_\varepsilon(p)$  contains some point  $t$ . Since  $B_\varepsilon(p) \cap \text{aff}(F) \subseteq F$ , it follows that  $t \in H \cap F$ . Let  $T$  be the line segment joining  $w$  and  $t$ . Since  $t \in H$  and  $w \in H'$  it follows that  $T \cap V$  contains exactly one point, say  $\bar{w}$ . Note that  $\bar{w} \neq t, w$ . Since  $w \notin S$  and  $t \in F$ , we have that  $\bar{w} \in \text{conv}(S \cup \{w\}) \cap V$  but  $\bar{w} \notin S$ .  $\square$

**2.2 Proof of Theorem 2.2** To simplify notation, given  $S \subseteq \mathbb{R}^n$ , we denote  $\mathbf{int}_V(S)$  simply by  $\mathbf{int}(S)$ .

The following standard result in lattice theory provides a useful equivalent definition of lattice (see Barvinok [7], p. 284 Theorem 1.4).

**THEOREM 2.3** *Let  $\Lambda$  be the additive group generated by vectors  $a_1, \dots, a_m \in \mathbb{R}^n$ . Then  $\Lambda$  is a lattice of the linear space  $\langle a_1, \dots, a_m \rangle$  if and only if there exists  $\varepsilon > 0$  such that  $\|y\| \geq \varepsilon$  for every  $y \in \Lambda \setminus \{0\}$ .*

In this paper we will only need the “only if” part of the statement, which is easy to prove (see [7], p. 281 problem 5). Theorem 2.3 implies the following result (see [7], p. 281 problem 3).

**COROLLARY 2.1** *Let  $\Lambda$  be a lattice of a linear space of  $\mathbb{R}^n$ . Then every bounded set in  $\mathbb{R}^n$  contains a finite number of points in  $\Lambda$ .*

Throughout this section,  $\Lambda$  will be a lattice of a linear space  $V$  of  $\mathbb{R}^n$ . The following lemma proves the “only if” part of Theorem 2.2 when  $S$  is bounded and full-dimensional.

**LEMMA 2.1** *Let  $S \subset V$  be a bounded maximal  $\Lambda$ -free convex set with  $\dim(S) = \dim(V)$ . Then  $S$  is a polytope with a point of  $\Lambda$  in the relative interior of each of its facets.*

**PROOF.** Since  $S$  is bounded, there exist integers  $L, U$  such that  $S$  is contained in the box  $B = \{x \in \mathbb{R}^n \mid L \leq x_i \leq U\}$ . For each  $y \in \Lambda \cap B$ , since  $S$  is convex there exists a closed half-space  $H^y$  of  $V$  such that  $S \subseteq H^y$  and  $y \notin \mathbf{int}(H^y)$ . By Corollary 2.1,  $B \cap \Lambda$  is finite, therefore  $\bigcap_{y \in B \cap \Lambda} H^y$  is a polyhedron. Thus

$P = \bigcap_{y \in B \cap \Lambda} H^y \cap B$  is a polytope and by construction  $\Lambda \cap \mathbf{int}(P) = \emptyset$ . Since  $S \subseteq B$  and  $S \subseteq H^y$  for every  $y \in B \cap \Lambda$ , it follows that  $S \subseteq P$ . By maximality of  $S$ ,  $S = P$ , therefore  $S$  is a polytope. We only need to show that  $S$  has a point of  $\Lambda$  in the relative interior of each of its facets. Let  $F_1, \dots, F_t$  be the facets of  $S$ , and let  $H_i = \{x \in V \mid \alpha_i x \leq \beta_i\}$  be the closed half-space defining  $F_i$ ,  $i = 1, \dots, t$ . Then  $S = \bigcap_{i=1}^t H_i$ . Suppose, by contradiction, that one of the facets of  $S$ , say  $F_t$ , does not contain a point of  $\Lambda$  in its relative interior. Given  $\varepsilon > 0$ , the polyhedron  $S' = \{x \in V \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \beta_t + \varepsilon\}$  is a polytope since it has the same recession cone as  $S$ . The polytope  $S'$  contains points of  $\Lambda$  in its interior by the maximality of  $S$ . By Corollary 2.1,  $\mathbf{int}(S')$  has a finite number of points in  $\Lambda$ , hence there exists one minimizing  $\alpha_t x$ , say  $z$ . By construction, the polytope  $S' = \{x \in V \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \alpha_t z\}$  does not contain any point of  $\Lambda$  in its interior and properly contains  $S$ , contradicting the maximality of  $S$ .  $\square$

We will also need the following famous theorem of Dirichlet.

**THEOREM 2.4 (DIRICHLET)** *Given real numbers  $\alpha_1, \dots, \alpha_n, \varepsilon$  with  $0 < \varepsilon < 1$ , there exist integers  $p_1, \dots, p_n$  and  $q$  such that*

$$\left| \alpha_i - \frac{p_i}{q} \right| < \frac{\varepsilon}{q}, \text{ for } i = 1, \dots, n, \text{ and } 1 \leq q \leq \varepsilon^{-1}. \quad (7)$$

The following is a consequence of Dirichlet's theorem.

**LEMMA 2.2** *Given  $y \in \Lambda$  and  $r \in V \setminus \{0\}$ , then for every  $\varepsilon > 0$  and  $\bar{\lambda} \geq 0$ , there exists a point of  $\Lambda \setminus \{y\}$  at distance less than  $\varepsilon$  from the half line  $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ .*

**PROOF.** First we show that, if the statement holds for  $\bar{\lambda} = 0$ , then it holds for arbitrary  $\bar{\lambda}$ . Given  $\varepsilon > 0$ , let  $Z$  be the set of points of  $\Lambda$  at distance less than  $\varepsilon$  from  $\{y + \lambda r \mid \lambda \geq 0\}$ . Suppose, by contradiction, that no point in  $Z$  has distance less than  $\varepsilon$  from  $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ . Then  $Z$  is contained in  $B_\varepsilon(0) + \{y + \lambda r \mid 0 \leq \lambda \leq \bar{\lambda}\}$ . By Corollary 2.1,  $Z$  is finite, thus there exists an  $\bar{\varepsilon} > 0$  such that every point in  $Z$  has distance greater than  $\bar{\varepsilon}$  from  $\{y + \lambda r \mid \lambda \geq 0\}$ , a contradiction. So we only need to show that, given  $\varepsilon > 0$ , there exists at least one point of  $\Lambda \setminus \{y\}$  at distance at most  $\varepsilon$  from  $\{y + \lambda r \mid \lambda \geq 0\}$ . We may assume  $\varepsilon < 1$ .

Without loss of generality, assume  $\|r\| = 1$ . Let  $m = \dim(V)$  and  $a_1, \dots, a_m$  be a basis of  $\Lambda$ . Then there exists  $\alpha \in \mathbb{R}^m$  such that  $r = \alpha_1 a_1 + \dots + \alpha_m a_m$ . Denote by  $A$  the matrix with columns  $a_1, \dots, a_m$ , and define  $\|A\| = \sup_{x: \|x\| \leq 1} \|Ax\|$  where, for a vector  $v$ ,  $\|v\|$  denotes the Euclidean norm of  $v$ . Choose  $\delta > 0$  such that  $\delta < 1$  and  $\delta \leq \varepsilon / (\|A\| \sqrt{m})$ . By Dirichlet's theorem, there exist  $p \in \mathbb{Z}^m$  and  $\lambda \geq 1$  such that

$$\left\| \alpha - \frac{p}{\lambda} \right\| = \sqrt{\sum_{i=1}^m \left| \alpha_i - \frac{p_i}{\lambda} \right|^2} \leq \frac{\delta \sqrt{m}}{\lambda} \leq \frac{\varepsilon}{\|A\| \lambda}.$$

Let  $z = Ap + y$ . Since  $p \in \mathbb{Z}^m$ , then  $z \in \Lambda$ . Note that  $p \neq 0$  since  $\|\alpha\| \geq \frac{\|A\alpha\|}{\|A\|} = \frac{\|r\|}{\|A\|} > \frac{\varepsilon}{\|A\| \lambda}$ , where the first inequality follows from the definition of  $\|A\|$  and the last one follows from the assumptions on  $\|r\|$ ,  $\varepsilon$  and  $\lambda$ . Therefore,  $z \in \Lambda \setminus \{y\}$ . Furthermore

$$\|(y + \lambda r) - z\| = \|\lambda r - Ap\| = \|A(\lambda \alpha - p)\| \leq \|A\| \|\lambda \alpha - p\| \leq \varepsilon.$$

$\square$

**LEMMA 2.3** *Let  $S$  be a  $\Lambda$ -free convex set, and let  $C = \text{rec}(S)$ . Then also  $S + \langle C \rangle$  is  $\Lambda$ -free.*

**PROOF.** Let  $r \in C$ ,  $r \neq 0$ . We only need to show that  $S + \langle r \rangle$  is  $\Lambda$ -free. Suppose there exists  $y \in \mathbf{int}(S + \langle r \rangle) \cap \Lambda$ . We show that  $y \in \mathbf{int}(S) + \langle r \rangle$ . Suppose not. Then  $(y + \langle r \rangle) \cap \mathbf{int}(S) = \emptyset$ , which implies that there is a hyperplane  $H$  separating the line  $y + \langle r \rangle$  and  $S + \langle r \rangle$ . This contradicts  $y \in \mathbf{int}(S + \langle r \rangle)$ . This shows  $y \in \mathbf{int}(S) + \langle r \rangle$ . Thus there exists  $\bar{\lambda}$  such that  $\bar{y} = y + \lambda r \in \mathbf{int}(S)$ , i.e. there exists  $\varepsilon > 0$  such that  $B_\varepsilon(\bar{y}) \cap V \subset S$ . Since  $y \in \Lambda$ , then  $y \notin \mathbf{int}(S)$ , and thus, since  $\bar{y} \in \mathbf{int}(S)$  and

$r \in C$ , we must have  $\bar{\lambda} > 0$ . Since  $r \in C$ , then  $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset S$ . Since  $y \in \Lambda$ , by Lemma 2.2 there exists  $z \in \Lambda$  at distance less than  $\varepsilon$  from the half line  $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ . Thus  $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$ , hence  $z \in \mathbf{int}(S)$ , a contradiction.  $\square$

Given a linear subspace  $L$  of  $\mathbb{R}^n$ , we denote by  $L^\perp$  the orthogonal complement of  $L$ . Given a set  $S \subseteq \mathbb{R}^n$ , the *orthogonal projection of  $S$  onto  $L^\perp$*  is the set

$$\text{proj}_{L^\perp}(S) = \{v \in L^\perp \mid v + w \in S \text{ for some } w \in L\}.$$

We will use the following result (see Barvinok [7], p. 284 problem 3).

LEMMA 2.4 *Given a  $\Lambda$ -subspace  $L$  of  $V$ , the orthogonal projection of  $\Lambda$  onto  $L^\perp$  is a lattice of  $L^\perp \cap V$ .*

LEMMA 2.5 *If a linear subspace  $L$  of  $V$  is not a  $\Lambda$ -subspace of  $V$ , then for every  $\varepsilon > 0$  there exists  $y \in \Lambda \setminus L$  at distance less than  $\varepsilon$  from  $L$ .*

PROOF. The proof is by induction on  $k = \dim(L)$ . Assume  $L$  is a linear subspace of  $V$  that is not a  $\Lambda$ -subspace, and let  $\varepsilon > 0$ . If  $k = 1$ , then, since the origin  $0$  is contained in  $\Lambda$ , by Lemma 2.2 there exists  $y \in \Lambda$  at distance less than  $\varepsilon$  from  $L$ . If  $y \in L$ , then  $L = \langle y \rangle$ , thus  $L$  is a  $\Lambda$ -subspace of  $V$ , contradicting our assumption.

Hence we may assume that  $k \geq 2$  and the statement holds for spaces of dimension  $k - 1$ .

*Case 1:  $L$  contains a nonzero vector  $r \in \Lambda$ . Let*

$$L' = \text{proj}_{\langle r \rangle^\perp}(L), \quad \Lambda' = \text{proj}_{\langle r \rangle^\perp}(\Lambda).$$

By Lemma 2.4,  $\Lambda'$  is a lattice of  $\langle r \rangle^\perp \cap V$ . Also,  $L'$  is not a lattice subspace of  $\langle r \rangle^\perp \cap V$  with respect to  $\Lambda'$ , because if there exists a basis  $a_1, \dots, a_{k-1}$  of  $L'$  contained in  $\Lambda'$ , then there exist scalars  $\mu_1, \dots, \mu_{k-1}$  such that  $a_1 + \mu_1 r, \dots, a_{k-1} + \mu_{k-1} r \in \Lambda$ , but then  $r, a_1 + \mu_1 r, \dots, a_{k-1} + \mu_{k-1} r$  is a basis of  $L$  contained in  $\Lambda$ , a contradiction. By induction, there exists a point  $y' \in \Lambda' \setminus L'$  at distance less than  $\varepsilon$  from  $L'$ . Since  $y' \in \Lambda'$ , there exists a scalar  $\mu$  such that  $y = y' + \mu r \in \Lambda$ , and  $y$  has distance less than  $\varepsilon$  from  $L$ .

*Case 2:  $L \cap \Lambda = \{0\}$ .* By Lemma 2.2, there exists a nonzero vector  $y \in \Lambda$  at distance less than  $\varepsilon$  from  $L$ . Since  $L$  does not contain any point in  $\Lambda$  other than the origin,  $y \notin L$ .  $\square$

LEMMA 2.6 *Let  $L$  be a linear subspace of  $V$  with  $\dim(L) = \dim(V) - 1$ , and let  $v \in V$ . Then  $v + L$  is a maximal  $\Lambda$ -free convex set if and only if  $L$  is not a lattice subspace of  $V$ .*

PROOF. ( $\Rightarrow$ ) Let  $S = v + L$  and assume that  $S$  is a maximal  $\Lambda$ -free convex set. Suppose by contradiction that  $L$  is a  $\Lambda$ -subspace. Then there exists a basis  $a_1, \dots, a_m$  of  $\Lambda$  such that  $a_1, \dots, a_{m-1}$  is a basis of  $L$ . Thus  $S = \{\sum_{i=1}^m x_i a_i \mid x_m = \beta\}$  for some  $\beta \in \mathbb{R}$ . Then,  $K = \{\sum_{i=1}^m x_i a_i \mid \lfloor \beta - 1 \rfloor \leq x_m \leq \lceil \beta \rceil\}$  strictly contains  $S$  and  $\mathbf{int}(K) \cap \Lambda = \emptyset$ , contradicting the maximality of  $S$ .

( $\Leftarrow$ ) Assume  $L$  is not a  $\Lambda$ -subspace of  $V$ . Since  $S = v + L$  is an affine hyperplane of  $V$ ,  $\mathbf{int}(S) = \emptyset$ , thus  $\mathbf{int}(S) \cap \Lambda = \emptyset$ , hence we only need to prove that  $S$  is maximal with such property. Suppose not, and let  $K$  be a maximal convex set in  $V$  such that  $\mathbf{int}(K) \cap \Lambda = \emptyset$  and  $S \subset K$ . Then by maximality  $K$  is closed. Let  $w \in K \setminus S$ . Since  $K$  is closed and convex,  $\overline{\text{conv}(\{w\} \cup S)} \subseteq K$ . Since  $\overline{\text{conv}(\{w\} \cup S)} = \overline{\text{conv}(\{w\} \cup (v + L))} = \text{conv}(\{v, w\}) + L$ , we have that  $K \supseteq \text{conv}(\{v, w\}) + L$ . Let  $\varepsilon$  be the distance between  $v + L$  and  $w + L$ , and  $\delta$  be the distance of  $\text{conv}(\{v, w\}) + L$  from the origin. By Lemma 2.5, since  $L$  is not a  $\Lambda$ -subspace of  $V$ , there exists a vector  $y \in \Lambda \setminus L$  at distance  $\bar{\varepsilon} < \varepsilon$  from  $L$ . Moreover, either  $y$  or  $-y$  has distance strictly less than  $\delta$  from  $\text{conv}(\{v, w\}) + L$ . We conclude that either  $(\lfloor \frac{\delta}{\bar{\varepsilon}} \rfloor + 1)y$  or  $-(\lfloor \frac{\delta}{\bar{\varepsilon}} \rfloor + 1)y$  is strictly between  $v + L$  and  $w + L$ , and therefore is in the interior of  $K$ . Since these two points are integer multiples of  $y \in \Lambda$ , this is a contradiction.  $\square$

We are now ready to prove Lovász's Theorem.

PROOF. [Proof of Theorem 2.2.] ( $\Leftarrow$ ) If  $S$  satisfies (ii), then by Lemma 2.6,  $S$  is a maximal  $\Lambda$ -free convex set. If  $S$  satisfies (i), then, since  $\mathbf{int}(S) \cap \Lambda = \emptyset$ , we only need to show that  $S$  is maximal. Suppose

not, and let  $K$  be a convex set in  $V$  such that  $\mathbf{int}(K) \cap \Lambda = \emptyset$  and  $S \subset K$ . Given  $y \in K \setminus S$ , there exists a hyperplane  $H$  separating  $y$  from  $S$  such that  $F = S \cap H$  is a facet of  $S$ . Since  $K$  is convex and  $S \subset K$ , then  $\text{conv}(S \cup \{y\}) \subseteq K$ . Since  $\dim(S) = \dim(V)$ ,  $F \subset S$  hence the  $\mathbf{relint}(F) \subset \mathbf{int}(K)$ . By assumption, there exists  $x \in \Lambda \cap \mathbf{relint}(F)$ , so  $x \in \mathbf{int}(K)$ , a contradiction.

( $\Rightarrow$ ) Let  $S$  be a maximal  $\Lambda$ -free convex set. We show that  $S$  satisfies either (i) or (ii). Observe that, by maximality,  $S$  must be closed.

If  $\dim(S) < \dim(V)$ , then  $S$  is contained in some affine hyperplane  $H$ . Since  $\mathbf{int}(H) = \emptyset$ , we have  $S = H$  by maximality of  $S$ , therefore  $S = v + L$  where  $v \in S$  and  $L$  is a hyperplane in  $V$ . By Lemma 2.6, (ii) holds.

Therefore we may assume that  $\dim(S) = \dim(V)$ . In particular, since  $S$  is convex,  $\mathbf{int}(S) \neq \emptyset$ . By Lemma 2.1, if  $S$  is bounded, (i) holds. Hence we may assume that  $S$  is unbounded. Let  $C$  be the recession cone of  $S$  and  $L$  the lineality space of  $S$ . By standard convex analysis,  $S$  is unbounded if and only if  $C \neq \{0\}$  (see for example Proposition 2.2.3 in [21]).

**Claim 1.**  $L = C$ .

By Lemma 2.3,  $S + \langle C \rangle$  is  $\Lambda$ -free. By maximality of  $S$  this implies that  $S = S + \langle C \rangle$ , hence  $\langle C \rangle \subseteq L$ . Since  $L \subseteq \langle C \rangle$ , it follows that  $L = C$ .  $\diamond$

Let  $P = \text{proj}_{L^\perp}(S)$  and  $\Lambda' = \text{proj}_{L^\perp}(\Lambda)$ . By Claim 1,  $S = P + L$  and  $P \subset L^\perp \cap V$  is a bounded set. Furthermore,  $\dim(S) = \dim(P) + \dim(L) = \dim(V)$  and  $\dim(P) = \dim(L^\perp \cap V)$ . Notice that  $\mathbf{int}(S) = \mathbf{relint}(P) + L$ , hence  $\mathbf{relint}(P) \cap \Lambda' = \emptyset$ . Furthermore  $P$  is inclusionwise maximal among the convex sets of  $L^\perp \cap V$  without points of  $\Lambda'$  in the relative interior: if not, given a convex set  $K \subseteq L^\perp \cap V$  strictly containing  $P$  and with no point of  $\Lambda'$  in its relative interior, we have  $S = P + L \subset K + L$ , and  $K + L$  does not contain any point of  $\Lambda$  in its interior, contradicting the maximality of  $S$ .

**Claim 2.**  $L$  is a  $\Lambda$ -subspace of  $V$ .

By contradiction, suppose  $L$  is not a  $\Lambda$ -subspace of  $V$ . Then, by Lemma 2.5, for every  $\varepsilon > 0$ , there exists a point in  $\Lambda \setminus L$  whose distance from  $L$  is at most  $\varepsilon$ . Therefore, its projection onto  $L^\perp$  is a point  $y \in \Lambda' \setminus \{0\}$  such that  $\|y\| < \varepsilon$ . Let  $V_\varepsilon$  be the linear subspace of  $L^\perp \cap V$  generated by the points in  $\{y \in \Lambda' \mid \|y\| < \varepsilon\}$ . Then  $\dim(V_\varepsilon) > 0$ .

Notice that, given  $\varepsilon' > \varepsilon'' > 0$ , then  $V_{\varepsilon'} \supseteq V_{\varepsilon''} \supset \{0\}$ , hence there exists  $\varepsilon_0 > 0$  such that  $V_\varepsilon = V_{\varepsilon_0}$  for every  $\varepsilon < \varepsilon_0$ . Let  $U = V_{\varepsilon_0}$ .

By definition,  $\Lambda'$  is dense in  $U$  (i.e. for every  $\varepsilon > 0$  and every  $x \in U$  there exists  $y \in \Lambda'$  such that  $\|x - y\| < \varepsilon$ ). Thus, since  $\mathbf{relint}(P) \cap \Lambda' = \emptyset$ , we also have  $\mathbf{relint}(P) \cap U = \emptyset$ . Since  $\dim(P) = \dim(L^\perp \cap V)$ , it follows that  $\mathbf{relint}(P) \cap (L^\perp \cap V) \neq \emptyset$ , so in particular  $U$  is a proper subspace of  $L^\perp \cap V$ .

Let  $Q = \text{proj}_{(L+U)^\perp}(P)$  and  $\Lambda'' = \text{proj}_{(L+U)^\perp}(\Lambda')$ . We show that  $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$ . Suppose not, and let  $y \in \mathbf{relint}(Q) \cap \Lambda''$ . Then,  $y + w \in \Lambda'$  for some  $w \in U$ . Furthermore, we claim that  $y + w' \in \mathbf{relint}(P)$  for some  $w' \in U$ . Indeed, suppose no such  $w'$  exists. Then  $(y + U) \cap (\mathbf{relint}(P) + U) = \emptyset$ . So there exists a hyperplane  $H$  in  $L^\perp \cap V$  separating  $y + U$  and  $P + U$ . Therefore the projection of  $H$  onto  $(L + U)^\perp$  separates  $y$  and  $Q$ , contradicting  $y \in \mathbf{relint}(Q)$ . Thus  $z = y + w' \in \mathbf{relint}(P)$  for some  $w' \in U$ . Since  $z \in \mathbf{relint}(P)$ , there exists  $\bar{\varepsilon} > 0$  such that  $B_{\bar{\varepsilon}}(z) \cap (L^\perp \cap V) \subset \mathbf{relint}(P)$ . Since  $\Lambda'$  is dense in  $U$  and  $y + w \in \Lambda'$ , it follows that  $\Lambda'$  is dense in  $y + U$ . Hence, since  $z \in y + U$ , there exists  $\bar{x} \in \Lambda'$  such that  $\|\bar{x} - z\| < \bar{\varepsilon}$ , hence  $\bar{x} \in \mathbf{relint}(P)$ , a contradiction. This shows  $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$ .

Finally, since  $\mathbf{relint}(Q) \cap \Lambda'' = \emptyset$ , then  $\mathbf{int}(Q + L + U) \cap \Lambda = \emptyset$ . Furthermore  $P \subseteq Q + U$ , therefore  $S \subseteq Q + L + U$ . By the maximality of  $S$ ,  $S = Q + L + U$  hence the lineality space of  $S$  contains  $L + U$ , contradicting the fact that  $L$  is the lineality space of  $S$  and  $U \neq \{0\}$ .  $\diamond$

Since  $L$  is a  $\Lambda$ -subspace of  $V$ ,  $\Lambda'$  is a lattice of  $L^\perp \cap V$  by Lemma 2.4. Since  $P$  is a bounded maximal  $\Lambda'$ -free convex set, it follows from Lemma 2.1 that  $P$  is a polytope with a point of  $\Lambda'$  in the relative

interior of each of its facets, therefore  $S = P + L$  has a point of  $\Lambda$  in the relative interior of each of its facets, and (i) holds.  $\square$

From the proof of Theorem 2.2 we get the following.

**COROLLARY 2.2** *Every  $\Lambda$ -free convex set of  $V$  is contained in some maximal  $\Lambda$ -free convex set of  $V$ .*

**PROOF.** Let  $S$  be a  $\Lambda$ -free convex set of  $V$ . If  $S$  is bounded, the proof of Lemma 2.1 shows that the corollary holds. If  $S$  is unbounded, Claim 1 in the proof of Theorem 2.2 shows that  $S + \langle C \rangle$  is  $\Lambda$ -free, where  $C$  is the recession cone of  $S$ . Hence we may assume that the lineality space  $L$  of  $S$  is equal to the recession cone of  $S$ . The projection  $P$  of  $S$  onto  $L^\perp$  is bounded. If  $L$  is a  $\Lambda$ -subspace, then  $\Lambda' = \text{proj}_{L^\perp} \Lambda$  is a lattice and  $P$  is  $\Lambda'$ -free, hence it is contained in a maximal  $\Lambda'$ -free convex set  $B$  of  $L^\perp \cap V$ , and  $B + L$  is a maximal  $\Lambda$ -free convex set of  $V$  containing  $S$ . If  $L$  is not a  $\Lambda$ -subspace, then we may define a linear subspace  $U$  of  $L^\perp \cap V$  and sets  $Q$  and  $\Lambda''$  as in the proof of Claim 2. Then proof of Claim 2 shows that  $Q$  is a bounded  $\Lambda''$ -free convex set of  $V \cap (L + U)^\perp$  and  $\Lambda''$  is a lattice, thus  $Q$  is contained in a maximal  $\Lambda''$ -free convex set  $B$  of  $V \cap (L + U)^\perp$ , and  $B + (L + U)$  is a maximal  $\Lambda$ -free convex set of  $V$  containing  $S$ .  $\square$

**3. Minimal Valid Inequalities** In this section we will prove Theorem 1.3. For ease of notation, we denote  $R_f(W)$  simply by  $R_f$  in this section. A linear function  $\Psi : \mathcal{W} \rightarrow \mathbb{R}$  is of the form

$$\Psi(s) = \sum_{r \in W} \psi(r) s_r, \quad s \in \mathcal{W} \quad (8)$$

for some  $\psi : W \rightarrow \mathbb{R}$ . Throughout the rest of the paper, capitalized Greek letters indicate linear functions from  $\mathcal{W}$  to  $\mathbb{R}$ , while the corresponding lowercase letters indicate functions from  $W$  to  $\mathbb{R}$  as defined in (8).

**DEFINITION 3.1** *A function  $\sigma : W \rightarrow \mathbb{R}$  is positively homogeneous if  $\sigma(\lambda r) = \lambda \sigma(r)$  for every  $r \in W$  and scalar  $\lambda \geq 0$ , and it is subadditive if  $\sigma(r^1 + r^2) \leq \sigma(r^1) + \sigma(r^2)$  for every  $r^1, r^2 \in W$ . The function  $\sigma$  is sublinear if it is positively homogeneous and subadditive.*

Note that if  $\sigma$  is sublinear, then  $\sigma(0) = 0$ . One can easily show that a function is sublinear if and only if it is positively homogeneous and convex. We also recall that convex functions are continuous on their domain, so if  $\sigma$  is sublinear it is also continuous [21].

**DEFINITION 3.2** *Inequality  $\sum_{r \in W} \psi(r) s_r \geq \alpha$  dominates inequality  $\sum_{r \in W} \psi'(r) s_r \geq \alpha$  if  $\psi(r) \leq \psi'(r)$  for all  $r \in W$ .*

**LEMMA 3.1** *Let  $\Psi(s) \geq \alpha$  be a valid linear inequality for  $R_f$ . Then  $\Psi(s) \geq \alpha$  is dominated by a valid linear inequality  $\Psi'(s) \geq \alpha$  for  $R_f$  such that  $\psi'$  is sublinear.*

*Proof:* We first prove the following.

**Claim 1.** *For every  $s \in \mathcal{W}$  such that  $\sum_{r \in W} r s_r = 0$  and  $s_r \geq 0$ ,  $r \in W$ , we have  $\sum_{r \in W} \psi(r) s_r \geq 0$ .*

Suppose not. Then there exists  $s \in \mathcal{W}$  such that  $\sum_{r \in W} r s_r = 0$ ,  $s_r \geq 0$  for all  $r \in W$  and  $\sum_{r \in W} \psi(r) s_r < 0$ . Let  $\bar{x}$  be an integral point in  $W$ . For any  $\lambda > 0$ , we define  $s^\lambda \in \mathcal{W}$  by

$$s_r^\lambda = \begin{cases} 1 + \lambda s_r & \text{for } r = \bar{x} - f \\ \lambda s_r & \text{otherwise.} \end{cases}$$

Since  $f + \sum_{r \in W} r s_r^\lambda = \bar{x}$ , it follows that  $s^\lambda$  is in  $R_f$ . Furthermore  $\sum_{r \in W} \psi(r) s_r^\lambda = \psi(\bar{x} - f) + \lambda(\sum_{r \in W} \psi(r) s_r)$ . Therefore  $\sum_{r \in W} \psi(r) s_r^\lambda$  goes to  $-\infty$  as  $\lambda$  goes to  $+\infty$ .  $\diamond$

We define, for all  $\bar{r} \in W$ ,

$$\psi'(\bar{r}) = \inf \left\{ \sum_{r \in W} \psi(r) s_r \mid \bar{r} = \sum_{r \in W} r s_r, s \in \mathcal{W}, s_r \geq 0 \text{ for all } r \in W \right\}.$$

By Claim 1,  $\sum_{r \in W} \psi(r)s_r \geq -\psi(-\bar{r})$  for all  $s \in \mathcal{W}$  such that  $\bar{r} = \sum_{r \in W} rs_r$  and  $s_r \geq 0$  for all  $r \in W$ . Thus the infimum in the above equation is finite and the function  $\psi'$  is well defined. Note also that  $\psi'(\bar{r}) \leq \psi(\bar{r})$  for all  $\bar{r} \in W$ , as follows by considering  $s \in \mathcal{W}$  defined by  $s_{\bar{r}} = 1$ ,  $s_r = 0$  for all  $r \in W$ ,  $r \neq \bar{r}$ .

**Claim 2.** *The function  $\psi'$  is sublinear*

Note first that  $\psi'(0) = 0$ . Indeed, Claim 1 implies  $\psi'(0) \geq 0$ , while choosing  $s_r = 0$  for all  $r \in W$  shows  $\psi'(0) \leq 0$ .

Next we show that  $\psi'$  is positively homogeneous. To prove this, let  $\bar{r} \in W$  and  $s \in \mathcal{W}$  such that  $\bar{r} = \sum_{r \in W} rs_r$  and  $s_r \geq 0$  for all  $r \in W$ . Let  $\gamma = \sum_{r \in W} \psi(r)s_r$ . For every  $\lambda > 0$ ,  $\lambda\bar{r} = \sum_{r \in W} r(\lambda s_r)$ ,  $\lambda s_r \geq 0$  for all  $r \in W$ , and  $\sum_{r \in W} \psi(r)(\lambda s_r) = \lambda\gamma$ . Therefore  $\psi'(\lambda\bar{r}) = \lambda\psi'(r)$ .

Finally, we show that  $\psi'$  is convex. Suppose by contradiction that there exist  $r', r'' \in W$  and  $0 < \lambda < 1$  such that  $\psi'(\lambda r' + (1-\lambda)r'') > \lambda\psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$  for some positive  $\epsilon$ . By definition of  $\psi'$ , there exist  $s', s'' \in \mathcal{W}$  such that  $r' = \sum_{r \in W} rs'_r$ ,  $r'' = \sum_{r \in W} rs''_r$ ,  $s'_r, s''_r \geq 0$  for all  $r \in W$ ,  $\sum_{r \in W} \psi(r)s'_r < \psi'(r') + \epsilon$  and  $\sum_{r \in W} \psi(r)s''_r < \psi'(r'') + \epsilon$ . Since  $\sum_{r \in W} r(\lambda s'_r + (1-\lambda)s''_r) = \lambda r' + (1-\lambda)r''$ , it follows that  $\psi'(\lambda r' + (1-\lambda)r'') \leq \sum_{r \in W} \psi(r)(\lambda s'_r + (1-\lambda)s''_r) < \lambda\psi'(r') + (1-\lambda)\psi'(r'') + \epsilon$ , a contradiction.  $\diamond$

**Claim 3.** *The inequality  $\sum_{r \in W} \psi(r)s_r \geq \alpha$  is valid for  $R_f$ .*

Suppose there exists  $\bar{s} \in R_f$  such that  $\sum_{r \in W} \psi'(r)\bar{s}_r \leq \alpha - \epsilon$  for some positive  $\epsilon$ . Let  $\{r^1, \dots, r^k\} = \{r \in W \mid \bar{s}_r > 0\}$ . For every  $i = 1, \dots, k$ , there exists  $s^i \in \mathcal{W}$  such that  $r^i = \sum_{r \in W} rs^i_r$ ,  $s^i_r \geq 0$ ,  $r \in W$ , and  $\sum_{r \in W} \psi(r)s^i_r < \psi'(r^i) + \epsilon/(k\bar{s}_{r^i})$ .

Let  $\tilde{s} = \sum_{i=1}^k \bar{s}_{r^i} s^i$ . Then

$$\sum_{r \in W} r\tilde{s}_r = \sum_{r \in W} \sum_{i=1}^k r\bar{s}_{r^i} s^i_r = \sum_{i=1}^k \bar{s}_{r^i} \sum_{r \in W} r s^i_r = \sum_{i=1}^k r^i \bar{s}_{r^i} = \sum_{r \in W} r\bar{s}_r,$$

hence  $\tilde{s} \in R_f$ . Therefore  $\sum_{r \in W} \psi(r)\tilde{s}_r \geq \alpha$  since  $\sum_{r \in W} \psi(r)s_r \geq \alpha$  is valid for  $R_f$ . Now

$$\begin{aligned} \sum_{r \in W} \psi(r)\tilde{s}_r &= \sum_{r \in W} \sum_{i=1}^k \psi(r)\bar{s}_{r^i} s^i_r = \sum_{i=1}^k \bar{s}_{r^i} \sum_{r \in W} \psi(r)s^i_r \\ &< \sum_{i=1}^k \bar{s}_{r^i} (\psi'(r^i) + \epsilon/(k\bar{s}_{r^i})) = \sum_{r \in W} \psi'(r)\bar{s}_r + \epsilon \leq \alpha, \end{aligned}$$

a contradiction.  $\square$

Recall the following definitions from the introduction.

**DEFINITION 3.3** *A valid inequality  $\sum_{r \in W} \psi(r)s_r \geq \alpha$  for  $R_f$  is minimal if it is not dominated by any valid linear inequality  $\sum_{r \in W} \psi'(r)s_r \geq \alpha$  for  $R_f$  such that  $\psi' \neq \psi$ .*

**DEFINITION 3.4** *Let  $V$  be the affine hull of  $(f + W) \cap \mathbb{Z}^q$ . Let  $C \in \mathbb{R}^{\ell \times q}$  and  $d \in \mathbb{R}^\ell$  be such that  $V = \{x \in f + W \mid Cx = d\}$ . Given two valid inequalities  $\sum_{r \in W} \psi(r)s_r \geq \alpha$  and  $\sum_{r \in W} \psi'(r)s_r \geq \alpha'$  for  $R_f(W)$ , we say that they are equivalent if there exist  $\rho > 0$  and  $\lambda \in \mathbb{R}^\ell$  such that  $\psi(r) = \rho\psi'(r) + \lambda^T Cr$  and  $\alpha = \rho\alpha' + \lambda^T(d - Cf)$ .*

**LEMMA 3.2** *Let  $\Psi(s) \geq \alpha$  and  $\Psi'(s) \geq \alpha'$  be two equivalent valid linear inequalities for  $R_f$ . (i) The function  $\psi$  is sublinear if and only if  $\psi'$  is sublinear.*

*(ii) Inequality  $\Psi(s) \geq \alpha$  is dominated by a minimal valid linear inequality if and only if  $\Psi'(s) \geq \alpha'$  is dominated by a minimal valid linear inequality. In particular,  $\Psi(s) \geq \alpha$  is minimal if and only if  $\Psi'(s) \geq \alpha'$  is minimal.*

PROOF. Since  $\Psi(s) \geq \alpha$  and  $\Psi'(s) \geq \alpha'$  are equivalent, by definition there exist  $\rho > 0$  and  $\lambda \in \mathbb{R}^\ell$ , such that  $\psi(r) = \rho\psi'(r) + \lambda^T Cr$  and  $\alpha = \rho\alpha' + \lambda^T(d - Cf)$ . This proves (i).

Point (ii) follows from the fact that, given a function  $\bar{\psi}'$  such that  $\bar{\psi}'(r) \leq \psi'(r)$  for every  $r \in W$ , then the function  $\bar{\psi}$  defined by  $\bar{\psi}(r) = \rho\bar{\psi}'(r) + \lambda^T Cr$ ,  $r \in W$ , satisfies  $\bar{\psi}(r) \leq \psi(r)$  for every  $r \in W$ . Furthermore  $\bar{\psi}(r) < \psi(r)$  if and only if  $\bar{\psi}'(r) < \psi'(r)$ .  $\square$

Given a nontrivial valid linear inequality  $\Psi(s) \geq \alpha$  for  $R_f$  such that  $\psi$  is sublinear, we consider the set

$$B_\psi = \{x \in f + W \mid \psi(x - f) \leq \alpha\}.$$

Since  $\psi$  is continuous,  $B_\psi$  is closed. Since  $\psi$  is convex,  $B_\psi$  is convex. Since  $\psi$  defines a valid inequality,  $B_\psi$  is lattice-free. Indeed the interior of  $B_\psi$  is  $\mathbf{int}(B_\psi) = \{x \in f + W : \psi(x - f) < \alpha\}$ . Its boundary is  $\mathbf{bd}(B_\psi) = \{x \in f + W : \psi(x - f) = \alpha\}$ , and its recession cone is  $\mathbf{rec}(B_\psi) = \{x \in f + W : \psi(x - f) \leq 0\}$ . Note that  $f$  is in the interior of  $B_\psi$  if and only if  $\alpha > 0$  and  $f$  is on the boundary if and only if  $\alpha = 0$ .

REMARK 3.1 Given a linear inequality of the form  $\Psi(s) \geq 1$  such that  $\psi(r) \geq 0$  for all  $r \in W$ ,

$$\psi(r) = \inf\{t > 0 \mid f + t^{-1}r \in B_\psi\}, \quad r \in W.$$

PROOF. Let  $r \in W$ . If  $\psi(r) > 0$ , let  $t$  be the minimum positive number such that  $f + t^{-1}r \in B_\psi$ . Then  $f + t^{-1}r \in \mathbf{bd}(B_\psi)$ , hence  $\psi(t^{-1}r) = 1$  and by positive homogeneity  $\psi(r) = t$ . If  $\psi(r) = 0$ , then  $r \in \mathbf{rec}(B_\psi)$ , hence  $f + t^{-1}r \in B_\psi$  for every  $t > 0$ , thus the infimum in the above equation is 0.  $\square$

This remark shows that, if  $\psi$  is nonnegative, then it is the *gauge* of the convex set  $B_\psi - f$  (see [21]).

Before proving Theorem 1.3, we need the following general theorem about sublinear functions. Let  $K$  be a closed, convex set in  $W$  with the origin in its interior. The *polar* of  $K$  is the set  $K^* = \{y \in W \mid ry \leq 1 \text{ for all } r \in K\}$ . Clearly  $K^*$  is closed and convex, and since  $0 \in \mathbf{int}(K)$ , it is well known that  $K^*$  is bounded. In particular,  $K^*$  is a compact set. Also, since  $0 \in K$ ,  $K^{**} = K$  (see [21] for example). Let

$$\hat{K} = \{y \in K^* \mid \exists x \in K \text{ such that } xy = 1\}. \quad (9)$$

Note that  $\hat{K}$  is contained in the relative boundary of  $K^*$ . Let  $\rho_K : W \rightarrow \mathbb{R}$  be defined by

$$\rho_K(r) = \sup_{y \in \hat{K}} ry, \quad \text{for all } r \in W. \quad (10)$$

It is easy to show that  $\rho_K$  is sublinear.

THEOREM 3.1 (BASU ET AL. [9]) *Let  $K \subset W$  be a closed convex set containing the origin in its interior. Then  $K = \{r \in W \mid \rho_K(r) \leq 1\}$ . Furthermore, for every sublinear function  $\sigma$  such that  $K = \{r \mid \sigma(r) \leq 1\}$ , we have  $\rho_K(r) \leq \sigma(r)$  for every  $r \in W$ .*

REMARK 3.2 *Let  $K \subset W$  be a polyhedron containing the origin in its interior. Let  $a_1, \dots, a_t \in W$  such that  $K = \{r \in W \mid a_i r \leq 1, i = 1, \dots, t\}$ . Then  $\rho_K(r) = \max_{i=1, \dots, t} a_i r$ .*

PROOF. The polar of  $K$  is  $K^* = \text{conv}\{0, a_1, \dots, a_t\}$  (see Theorem 9.1 in Schrijver [24]). Furthermore,  $\hat{K}$  is the union of all the facets of  $K^*$  that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{i=1, \dots, t} a_i r$$

for all  $r \in W$ .  $\square$

REMARK 3.3 *Let  $B$  be a closed lattice-free convex set in  $f + W$  with  $f$  in its interior, and let  $K = B - f$ . Then the inequality  $\sum_{r \in W} \rho_K(r) s_r \geq 1$  is valid for  $R_f$ .*

*Proof:* Let  $s \in R_f$ . Then  $x = f + \sum_{r \in W} r s_r$  is integral, therefore  $x \notin \mathbf{int}(B)$  because  $B$  is lattice-free. By Theorem 3.1,  $\rho_K(x - f) \geq 1$ . Thus

$$1 \leq \rho_K\left(\sum_{r \in W} r s_r\right) \leq \sum_{r \in W} \rho_K(r s_r) \leq \sum_{r \in W} \rho_K(r) s_r,$$

where the second inequality follows from the subadditivity of  $\rho_K$  and the last from the positive homogeneity.  $\square$

LEMMA 3.3 *Given a maximal lattice-free convex set  $B$  of  $f + W$  containing  $f$  in its interior,  $\Psi_B(s) \geq 1$  is a minimal valid inequality for  $R_f$ .*

PROOF. Let  $\Psi(s) \geq 1$  be a valid linear inequality for  $R_f$  such that  $\psi(r) \leq \psi_B(r)$  for all  $r \in W$ . Then  $B_\psi \supset B$  and  $B_\psi$  is lattice-free. By maximality of  $B$ ,  $B = B_\psi$ . By Theorem 3.1 and Remark 3.2,  $\psi_B(r) \leq \psi(r)$  for all  $r \in W$ , proving  $\psi = \psi_B$ .  $\square$

PROOF. [Proof of Theorem 1.3]

Let  $\Psi(s) \geq \alpha$  be a nontrivial valid linear inequality for  $R_f$ . By Lemma 3.1, we may assume that  $\psi$  is sublinear.

**Claim 1** *If  $\text{int}(B_\psi) \cap V = \emptyset$ , then  $\Psi(s) \geq \alpha$  is trivial.*

Suppose  $\text{int}(B_\psi) \cap V = \emptyset$  and let  $s \in \mathcal{V}$  such that  $s_r \geq 0$  for every  $r \in W$ . Let  $x = f + \sum_{r \in W} r s_r$ . Since  $s \in \mathcal{V}$ ,  $x \in V$ , so  $x \notin \text{int}(B_\psi)$ . This implies

$$\alpha \leq \psi(x - f) = \psi\left(\sum_{r \in W} r s_r\right) \leq \sum_{r \in W} \psi(r) s_r = \Psi(s),$$

where the last inequality follows from the sublinearity of  $\psi$ .  $\diamond$

**Claim 2** *If  $f \in V$  and  $\alpha \leq 0$ , then  $\text{int}(B_\psi) \cap V = \emptyset$ .*

Suppose  $f \in V$ ,  $\alpha \leq 0$  but  $\text{int}(B_\psi) \cap V \neq \emptyset$ . Then  $\dim(\text{int}(B_\psi) \cap V) = \dim(V)$ , hence  $\text{int}(B_\psi) \cap V$  contains a set  $X$  of  $\dim(V) + 1$  affinely independent points. For every  $x \in X$  and every  $\lambda > 0$ ,  $\psi(\lambda(x - f)) = \lambda\psi(x - f) < 0$ , where the last inequality is because  $x \in \text{int}(B_\psi)$ . Hence the set  $\Gamma = f + \text{cone}\{x - f \mid x \in X\}$  is contained in  $\text{int}(B_\psi)$ . Since  $\Gamma$  has dimension equal to  $\dim(V)$  and  $V$  is the convex hull of its integral points,  $\Gamma \cap \mathbb{Z}^q \neq \emptyset$ , contradicting the fact that  $B_\psi$  has no integral point in its interior.  $\diamond$

**Claim 3** *If  $f \notin V$ , then there exists a valid linear inequality  $\Psi'(s) \geq 1$  for  $R_f$  equivalent to  $\Psi(s) \geq \alpha$ .*

Since  $f \notin V$ ,  $Cf \neq d$ , hence there exists a row  $c_i$  of  $C$  such that  $d_i - c_i f \neq 0$ . Let  $\lambda = (1 - \alpha)(d_i - c_i f)^{-1}$ , and define  $\psi'(r) = \psi(r) + \lambda c_i r$  for every  $r \in W$ . The inequality  $\Psi'(s) \geq 1$  is equivalent to  $\Psi(s) \geq \alpha$ .  $\diamond$

Thus, by Claims 1, 2 and 3 there exists a valid linear inequality  $\Psi'(s) \geq 1$  for  $R_f$  equivalent to  $\Psi(s) \geq \alpha$ . By Lemma 3.2,  $\psi'$  is sublinear and  $\Psi(s) \geq \alpha$  is dominated by a minimal valid linear inequality if and only if  $\Psi'(s) \geq 1$  is dominated by a minimal valid linear inequality. Therefore we only need to consider valid linear inequalities of the form  $\Psi(s) \geq 1$  where  $\psi$  is sublinear. In particular the set  $B_\psi = \{x \in W \mid \psi(x - f) \leq 1\}$  contains  $f$  in its interior.

Let  $K = \{r \in W \mid \psi(r) \leq 1\}$ , and let  $\hat{K}$  be defined as in (9).

**Claim 4** *The inequality  $\sum_{r \in W} \rho_K(r) s_r \geq 1$  is valid for  $R_f$  and  $\psi(r) \geq \rho_K(r)$  for all  $r \in W$ .*

Note that  $B_\psi = f + K$ . Thus, by Remark 3.3,  $\sum_{r \in W} \rho_K(r) s_r \geq 1$  is valid for  $R_f$ . Since  $\psi$  is sublinear, it follows from Theorem 3.1 that  $\rho_K(r) \leq \psi(r)$  for every  $r \in W$ .  $\diamond$

By Claim 4, since  $\rho_K$  is sublinear, we may assume that  $\psi = \rho_K$ .

**Claim 5** *There exists a valid linear inequality  $\Psi'(s) \geq 1$  for  $R_f$  dominating  $\Psi(s) \geq 1$  such that  $\psi'$  is sublinear,  $B_{\psi'}$  is a polyhedron, and  $\text{rec}(B_{\psi'} \cap V) = \text{lin}(B_{\psi'} \cap V)$ .*

Since  $B_\psi$  is a lattice-free convex set, it is contained in some maximal lattice-free convex set  $S$  by Corollary 2.2. The set  $S$  satisfies one of the statements (i)-(iii) of Theorem 2.1. By Claim 1,  $\text{int}(S) \cap V \neq \emptyset$ ,

hence case (iii) does not apply. Case (ii) does not apply because  $\dim(S) = \dim(B_\psi) = \dim(W)$ . Therefore case (i) applies. Thus  $S$  is a polyhedron and  $S \cap V$  is a maximal lattice-free convex set in  $V$ . In particular, by Theorem 2.2,  $\text{rec}(S \cap V) = \text{lin}(S \cap V)$ . Since  $S$  is a polyhedron containing  $f$  in its interior, there exists  $A \in \mathbb{R}^{t \times q}$  and  $b \in \mathbb{R}^t$  such that  $b_i > 0$ ,  $i = 1, \dots, t$ , and  $S = \{x \in f + W \mid A(x - f) \leq b\}$ . Without loss of generality, we may assume that  $\sup_{x \in B_\psi} a_i(x - f) = 1$  where  $a_i$  denotes the  $i$ th row of  $A$ ,  $i = 1, \dots, t$ . By our assumption,  $\sup_{r \in K} a_i r = 1$ . Therefore  $a_i \in K^*$ , since  $a_i r \leq 1$  for all  $r \in K$ . Furthermore  $a_i \in \text{cl}(\hat{K})$ , since  $\sup_{r \in K} a_i r = 1$ .

Let  $\bar{S} = \{x \in f + W \mid A(x - f) \leq e\}$ , where  $e$  denotes the vector of all ones. Then  $B_\psi \subseteq \bar{S} \subseteq S$ . Let  $Q = \{r \in W \mid Ar \leq e\}$ . By Remark 3.2,  $\rho_Q(r) = \max_{i=1, \dots, t} a_i r$  for all  $r \in W$ . Since  $\bar{S} \subseteq S$ ,  $\bar{S}$  is lattice-free, by Remark 3.3 the inequality  $\sum_{r \in W} \rho_Q(r) s_r \geq 1$  is valid for  $R_f$ . Furthermore, since  $\{a_1, \dots, a_t\} \subset \text{cl}(\hat{K})$ , by Claim 4 we have

$$\psi(r) = \sup_{y \in \hat{K}} yr \geq \max_{i=1, \dots, t} a_i r = \rho_Q(r)$$

for all  $r \in W$ . Let  $\psi' = \rho(Q)$ . Note that  $B_{\psi'} = \bar{S}$ . So,  $\text{rec}(B_{\psi'}) = \text{rec}(\bar{S}) = \{r \in W \mid Ar \leq 0\} = \text{rec}(S)$ . Since  $\text{rec}(S \cap V) = \text{lin}(S \cap V)$ , then  $\text{rec}(B_{\psi'} \cap V) = \text{lin}(B_{\psi'} \cap V)$ .  $\diamond$

By Claim 5, we may assume that  $B_\psi = \{x \in f + W \mid A(x - f) \leq e\}$ , where  $A \in \mathbb{R}^{t \times q}$  and  $e$  is the vector of all ones, and that  $\text{rec}(B_\psi \cap V) = \text{lin}(B_\psi \cap V)$ . Let  $a_1, \dots, a_t$  denote the rows of  $A$ . By Claim 4 and Remark 3.2,

$$\psi(r) = \max_{i=1, \dots, t} a_i r, \quad \text{for all } r \in W. \quad (11)$$

Let  $G$  be a matrix such that  $W = \{r \in \mathbb{R}^q \mid Gr = 0\}$ .

**Claim 6** *There exists  $\lambda \in \mathbb{R}^\ell$  such that  $\psi(r) + \lambda^T Cr \geq 0$  for all  $r \in W$ .*

Given  $\lambda \in \mathbb{R}^\ell$ , then by (11)  $\psi(r) + \lambda^T Cr \geq 0$  for every  $r \in W$  if and only if  $\min_{r \in W} (\max_{i=1, \dots, t} a_i r + \lambda^T Cr) = 0$ . The latter holds if and only if

$$0 = \min\{z + \lambda^T Cr \mid ez - Ar \geq 0, Gr = 0\}.$$

By LP duality, this holds if and only if the following system is feasible

$$\begin{aligned} ey &= 1 \\ A^T y + C^T \lambda - G^T \mu &= 0 \\ y &\geq 0. \end{aligned}$$

Clearly the latter is equivalent to

$$\begin{aligned} A^T y + C^T \lambda - G^T \mu &= 0 \\ y &\geq 0, y \neq 0. \end{aligned} \quad (12)$$

Note that  $\text{rec}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar \leq 0, Cr = 0, Gr = 0\}$  and  $\text{lin}(B_\psi \cap V) = \{r \in \mathbb{R}^q \mid Ar = 0, Cr = 0, Gr = 0\}$ . Since  $\text{rec}(B_\psi \cap V) = \text{lin}(B_\psi \cap V)$ , the system

$$\begin{aligned} Ar &\leq 0 \\ Cr &= 0 \\ Gr &= 0 \\ e^T Ar &= -1 \end{aligned}$$

is infeasible. By Farkas Lemma, this is the case if and only if there exists  $\gamma \geq 0$ ,  $\lambda$ ,  $\tilde{\mu}$ , and  $\tau$  such that

$$A^T \gamma + C^T \lambda + G^T \tilde{\mu} + A^T e \tau = 0, \quad \tau > 0.$$

If we let  $y = \gamma + e\tau$  and  $\mu = -\tilde{\mu}$ , then  $(y, \lambda, \mu)$  satisfies (12). By the previous argument,  $\lambda$  satisfies the statement of the claim.  $\diamond$

Let  $\lambda$  as in Claim 6, and let  $\psi'$  be the function defined by  $\psi'(r) = \psi(r) + \lambda^T Cr$  for all  $r \in W$ . So  $\psi'(r) \geq 0$  for every  $r \in W$ . Let  $\alpha' = 1 + \lambda^T(d - Cf)$ . Then the inequality  $\Psi'(s) \geq \alpha'$  is valid for  $R_f$  and it is equivalent to  $\Psi(s) \geq \alpha$ . If  $\alpha' \leq 0$ , then  $\Psi'(s) \geq \alpha'$  is trivial. Thus  $\alpha' > 0$ . Let  $\rho = 1/\alpha'$  and let  $\psi'' = \rho\psi'$ . Then  $\Psi''(s) \geq 1$  is equivalent to  $\Psi(s) \geq 1$ . By Lemma 3.2(i),  $\psi''$  is sublinear.

Let  $B$  be a maximal lattice-free convex set of  $f + W$  containing  $B_{\psi''}$ . Such a set  $B$  exists by Corollary 2.2.

**Claim 7**  $\psi''(r) \geq \psi_B(r)$  for all  $r \in W$ .

Let  $r \in \text{rec}(B_{\psi''})$ . Since  $\psi''$  is nonnegative,  $\psi''(r) = 0$ . Since  $\text{rec}(B_{\psi''}) \subseteq \text{rec}(B)$ ,  $\psi_B(r) \leq 0 = \psi''(r)$ . Let  $r \notin \text{rec}(B_{\psi''})$ . Then  $f + \tau r \in \text{bd}(B_{\psi''})$  for some  $\tau > 0$ , hence  $\psi''(\tau r) = 1$  and, by positive homogeneity,  $\psi''(r) = \tau^{-1}$ . Because  $B_{\psi''} \subset B$ ,  $f + \tau r \in B$ . Since  $B = \{x \in f + W \mid \psi_B(x - f) \leq 1\}$ , it follows that  $\psi_B(\tau r) \leq 1$ , implying  $\psi_B(r) \leq \tau^{-1} = \psi''(r)$ .  $\diamond$

Claim 7 shows that  $\Psi''(s) \geq 1$  is dominated by  $\Psi_B(s) \geq 1$ , which is minimal by Lemma 3.3. By Lemma 3.2(ii),  $\Psi(s) \geq 1$  is dominated by a minimal valid linear inequality which is equivalent to  $\Psi_B(s) \geq 1$ .  $\square$

*Example.* We illustrate the end of the proof in an example. Suppose  $W = \{x \in \mathbb{R}^3 \mid x_2 + \sqrt{2}x_3 = 0\}$ , and let  $f = (\frac{1}{2}, 0, 0)$ . Note that  $f + W = W$ . All integral points in  $W$  are of the form  $(k, 0, 0)$ ,  $k \in \mathbb{Z}$ , hence  $V = \{x \in W \mid x_2 = 0\}$ . Thus  $\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r \in W} r_2 s_r = 0\}$ .

Consider the function  $\psi : W \rightarrow \mathbb{R}$  defined by  $\psi(r) = \max\{-4r_1 - 4r_2, 4r_1 - 4r_2\}$ . The set  $B_\psi = \{x \in W \mid -4(x_1 - \frac{1}{2}) - 4x_2 \leq 1, 4(x_1 - \frac{1}{2}) - 4x_2 \leq 1\}$  does not contain any integral point, hence  $\Psi(s) \geq 1$  is valid for  $R_f$ . Note that  $B_\psi$  is not maximal (see Figure 2).

Setting  $\lambda = 4$  in Claim 6, let  $\psi'(r) = \psi(r) + \lambda r_2$  for all  $r \in W$ . Note that  $\psi'(r) = \max\{-4r_1, 4r_1\} \geq 0$  for all  $r \in W$ . The set  $B_{\psi'} = \{x \in W \mid -4(x_1 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{2}) \leq 1\}$  is contained in the maximal lattice-free convex set  $B = \{x \in W \mid -2(x_1 - \frac{1}{2}) \leq 1, 2(x_1 - \frac{1}{2}) \leq 1\}$ , hence  $\psi'$  is pointwise larger than the function  $\psi_B$  defined by  $\psi_B(r) = \max\{-2r_1, 2r_1\}$  and  $\Psi_B(s) \geq 1$  is valid for  $R_f$ . This completes the illustration of the proof.

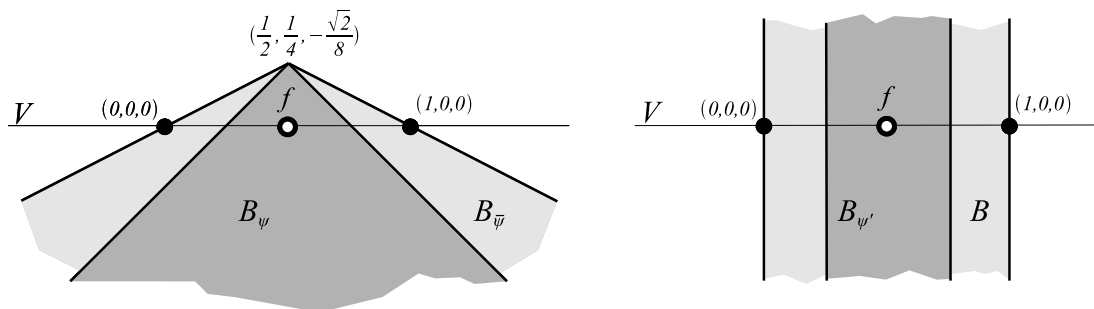


Figure 2: Lattice-free sets in the 2-dimensional space  $W$ .

Note that  $\Psi_B(s) \geq 1$  does not dominate  $\Psi(s) \geq 1$ . However the inequality  $\Psi_B(s) \geq 1$  is equivalent to a valid inequality  $\bar{\Psi}(s) \geq 1$  which dominates  $\Psi(s) \geq 1$ . We show how to construct  $\psi$  in our example. The function  $\bar{\psi}$  is defined by  $\bar{\psi}(r) = \psi(r) - \lambda r_2$  for all  $r \in W$  is pointwise smaller than  $\psi$  and  $\bar{\Psi}(s) \geq 1$  is valid for  $R_f$ . Moreover,  $B_{\bar{\psi}} = \{x \in W \mid -2(x_1 - \frac{1}{2}) - 4x_1 \leq 1, 2(x_1 - \frac{1}{2}) - 4x_1 \leq 1\}$  is a maximal lattice-free convex set containing  $B_\psi$ . Note that the recession cones of  $B_\psi$  and  $B_{\bar{\psi}}$  are full dimensional, hence  $\psi$  and  $\bar{\psi}$  take negative values on elements of the recession cone. For example  $\psi(0, -1, \frac{1}{\sqrt{2}}) = \bar{\psi}(0, -1, \frac{1}{\sqrt{2}}) = -4$ . The recession cones of  $B_{\psi'}$  and  $B$  coincide and are not full dimensional, thus  $\psi'(0, -1, \frac{1}{\sqrt{2}}) = \psi_B(0, -1, \frac{1}{\sqrt{2}}) = 0$ , since the vector  $(0, -1, \frac{1}{\sqrt{2}})$  is in the recession cone of  $B$ .

**4. The intersection of all minimal inequalities** In this section we prove Theorem 1.4. First we need the following.

**LEMMA 4.1** *Let  $\psi : W \rightarrow \mathbb{R}$  be a continuous function that is positively homogeneous. Then the function  $\Psi : \mathcal{W} \rightarrow \mathbb{R}$ , defined by  $\Psi(s) = \sum_{r \in W} \psi(r) s_r$ , is continuous with respect to  $(\mathcal{W}, \|\cdot\|_H)$ .*

*Proof:* Define  $\gamma = \sup\{|\psi(r)| : r \in W, \|r\| = 1\}$ . Since the set  $\{r \in R_f(W) : \|r\| = 1\}$  is compact and  $\psi$  is continuous,  $\gamma$  is well defined (that is, it is finite). Given  $s, s' \in \mathcal{W}$ , we will show  $|\Psi(s') - \Psi(s)| \leq$

$\gamma\|s' - s\|_H$ , which implies that  $\Psi$  is continuous. Indeed

$$\begin{aligned}
|\Psi(s') - \Psi(s)| &= \left| \sum_{r \in W} \psi(r)(s'_r - s_r) \right| \\
&\leq \sum_{r \in W} |\psi(r)| |s'_r - s_r| \\
&= \sum_{r \in W: \|r\|=1} \sum_{\alpha > 0} |\psi(\alpha r)| |s'_{\alpha r} - s_{\alpha r}| \\
&= \sum_{r \in W: \|r\|=1} |\psi(r)| \sum_{\alpha > 0} \alpha |s'_{\alpha r} - s_{\alpha r}| \quad (\text{by positive homogeneity of } \psi) \\
&\leq \gamma \sum_{r \in W: \|r\|=1} \sum_{\alpha > 0} \alpha |s'_{\alpha r} - s_{\alpha r}| \\
&= \gamma \sum_{r \in W \setminus \{0\}} \|r\| |s'_r - s_r| \leq \gamma \|s' - s\|_H
\end{aligned}$$

□

PROOF. [Proof of Theorem 1.4] “ $\subseteq$ ” By Lemma 4.1,  $\Psi_B$  is continuous in  $(\mathcal{W}, \|\cdot\|_H)$  for every  $B \in \mathcal{B}_W$ , therefore  $\{s \in \mathcal{W} : \Psi_B(s) \geq 1\}$  is a closed half-space of  $(\mathcal{W}, \|\cdot\|_H)$ . It is immediate to show that also  $\{s \in \mathcal{W} : s_r \geq 0, r \in W\}$  is a closed set in  $(\mathcal{W}, \|\cdot\|_H)$ . Since  $\mathcal{V} = \{s \in \mathcal{W} \mid \sum_{r \in W} (Cr)s_r = d - Cf\}$ , and since for each row  $c_i$  of  $C$  the function  $r \mapsto c_i r$  is positive homogeneous, then by Lemma 4.1  $\mathcal{V}$  is also closed. Thus

$$\{s \in \mathcal{V} : \Psi_B(s) \geq 1, B \in \mathcal{B}_W; s_r \geq 0, r \in W\}$$

is an intersection of closed sets, and is therefore a closed set of  $(\mathcal{W}, \|\cdot\|_H)$ . Thus, since it contains  $\text{conv}(R_f(W))$ , it also contains  $\overline{\text{conv}}(R_f(W))$ .

“ $\supseteq$ ” We only need to show that, for every  $\bar{s} \in \mathcal{V}$  such that  $\bar{s} \notin \overline{\text{conv}}(R_f(W))$  and  $\bar{s}_r \geq 0$  for every  $r \in W$ , there exists  $B \in \mathcal{B}_W$  such that  $\sum_{r \in W} \psi_B(r)\bar{s}_r < 1$ .

The theorem of Hahn-Banach implies the following.

*Given a closed convex set  $A$  in  $(\mathcal{W}, \|\cdot\|_H)$  and a point  $b \notin A$ , there exists a continuous linear function  $\Psi : \mathcal{W} \rightarrow \mathbb{R}$  that strictly separates  $A$  and  $b$ , i.e. for some  $\alpha \in \mathbb{R}$ ,  $\Psi(a) \geq \alpha$  for every  $a \in A$ , and  $\Psi(b) < \alpha$ .*

Therefore, there exists a linear function  $\Psi : \mathcal{W} \rightarrow \mathbb{R}$  such that  $\Psi(\bar{s}) < \alpha$  and  $\Psi(s) \geq \alpha$  for every  $s \in \overline{\text{conv}}(R_f)$ . By the first part of Theorem 1.3, we may assume that  $\Psi(s) \geq \alpha$  is a nontrivial minimal valid linear inequality. By the second part of Theorem 1.3, this inequality is equivalent to an inequality of the form  $\sum_{r \in W} \psi_B(r)s_r \geq 1$  for some maximal lattice-free convex set  $B$  of  $W$  with  $f$  in its interior. □

**5. Proof of Theorem 1.5** By Theorem 2.2,  $L := \text{lin}(B)$  is a rational space and  $P$  is a polytope. Also, by construction,  $\psi_B(r^j) = 1$  for  $j = 1, \dots, k$ , and  $\psi_B(r^j) = 0$  for  $j = k+1, \dots, k+h$ . Thus  $\sum_{j=1}^k s_j \geq 1$  is a valid inequality for  $R_f(r^1, \dots, r^{k+h})$ . We recall that, by Theorem 1.3,  $\sum_{r \in \mathbb{R}^q} \psi_B(r)s_r \geq 1$  is a minimal valid inequality for  $R_f(\mathbb{R}^q)$ .

We first show that  $R_f(r^1, \dots, r^{k+h})$  is nonempty. Since  $r^{k+1}, \dots, r^{k+h}$  are rational, there exists a positive integer  $N$  such that  $Nr^j$  is integral for  $j = k+1, \dots, k+h$ . Since  $f \in \text{int}(B)$ , it follows that, for every  $r \in \mathbb{R}^q$ , there exists  $s \in \mathbb{R}^{k+h}$  such that  $r = \sum_{j=1}^{k+h} r^j s_j$  and  $s_j \geq 0$ ,  $j = 1, \dots, k$ . Thus, given  $\bar{x} \in \mathbb{Z}^n$ , there exists  $\bar{s}$  such that  $\bar{x} - f = \sum_{j=1}^{k+h} r^j \bar{s}_j$  and  $\bar{s}_j \geq 0$ ,  $j = 1, \dots, k$ . Let  $\lambda$  be a positive integer such that  $\bar{s} + \lambda N \sum_{j=k+1}^{k+h} e^j \geq 0$ , where  $e_j$  denotes the  $j$ th unit vector in  $\mathbb{R}^{k+h}$ . Then  $\bar{s} + \lambda N \sum_{j=k+1}^{k+h} e^j \geq 0 \in R_f(r^1, \dots, r^{k+h})$ .

“ $\Leftarrow$ ” Let us assume that  $\sum_{j=1}^k s_j \geq 1$  is an extreme inequality for  $R_f(r^1, \dots, r^{k+h})$ . We show that  $\sum_{r \in \mathbb{R}^q} \psi_B(r)s_r \geq 1$  is extreme for  $R_f(\mathbb{R}^q)$ .

Let  $\sum_{r \in \mathbb{R}^q} \psi_i(r) s_r \geq 1$ ,  $i = 1, 2$ , be valid inequalities for  $R_f(\mathbb{R}^q)$  such  $\psi_B \geq \frac{1}{2}(\psi_1 + \psi_2)$ . We will show that  $\psi_B = \psi_1 = \psi_2$ . Since  $\psi_B$  is minimal and  $\psi_B \geq \frac{1}{2}(\psi_1 + \psi_2)$ , then  $\psi_1, \psi_2$  are both minimal. Thus, given  $B_i = \{x \in \mathbb{R}^q : \psi_i(x - f) \leq 1\}$ ,  $i = 1, 2$ ,  $B_1$  and  $B_2$  are maximal lattice-free convex sets. Furthermore, since  $\sum_{j=1}^k s_j \geq 1$  is extreme for  $R_f(r^1, \dots, r^{k+h})$ , then  $\psi_1(r^j) = \psi_2(r^j) = \psi_B(r^j)$  for  $j = 1, \dots, k + h$ . In particular,  $\psi_1(r^j) = \psi_2(r^j) = 1$  for  $j = 1, \dots, k$  and  $\psi_1(r^j) = \psi_2(r^j) = 0$  for  $j = k + 1, \dots, k + h$ . Hence  $B_1$  and  $B_2$  contain  $B$ , since they contain the vertices of  $P$  and their lineality space contains  $r^{k+1}, \dots, r^{k+h}$ . By the maximality of  $B$ ,  $B_1 = B_2 = B$ , therefore  $\psi_B = \psi_1 = \psi_2$ , proving that  $\sum_{r \in \mathbb{R}^q} \psi_B(r) s_r \geq 1$  is extreme.

“ $\Rightarrow$ ” Let us assume that  $\sum_{r \in \mathbb{R}^q} \psi_B(r) s_r \geq 1$  is an extreme inequality for  $R_f(\mathbb{R}^q)$ . We prove that  $\sum_{j=1}^k s_j \geq 1$  is an extreme inequality for  $R_f(r^1, \dots, r^{k+h})$ .

Let  $\alpha, \beta \in \mathbb{R}^{k+h}$  be vectors such that  $\alpha s \geq 1$  and  $\beta s \geq 1$  are valid for  $R_f(r^1, \dots, r^{k+h})$ , and  $\frac{1}{2}(\alpha_j + \beta_j) \leq \psi_B(r^j)$ ,  $j = 1, \dots, k + h$ . We will show that it must follow that  $\alpha_j = \beta_j = \psi_B(r^j)$  for  $j = 1, \dots, k + h$ .

We first observe that, for  $j = k + 1, \dots, k + h$ ,  $\alpha_j = \beta_j = 0$ . If not, since  $\frac{1}{2}(\alpha_j + \beta_j) \leq \psi_B(r^j) = 0$  for  $j = k + 1, \dots, k + h$ , then we may assume that  $\alpha_j < 0$  for some  $j \in \{k + 1, \dots, k + h\}$ . Given  $\bar{s} \in R_f(r^1, \dots, r^{k+h})$  it now follows that  $\bar{s} + \lambda N e^j \in R_f(r^1, \dots, r^{k+h})$  for every positive integer  $\lambda$ . However,  $\lim_{\lambda \rightarrow +\infty} \alpha(\bar{s} + \lambda N e^j) = \alpha \bar{s} + \lim_{\lambda \rightarrow +\infty} \lambda N \alpha_j = -\infty$ , contradicting the fact that  $\alpha s \geq 1$  is valid for  $R_f(r^1, \dots, r^{k+h})$ .

Define, for every  $r \in \mathbb{R}^q$ ,

$$\psi^\alpha(r) = \min\{\alpha s \mid \sum_{j=1}^{k+h} r^j s_j = r, s_j \geq 0, j = 1, \dots, k\}. \tag{13}$$

Note that, for every  $r \in \mathbb{R}^q$ , the above linear program is feasible. We also observe that, for every  $\bar{x} \in \mathbb{Z}^q$ ,  $\psi^\alpha(\bar{x} - f) \geq 1$ . Indeed, given  $\bar{s} \in \mathbb{R}^{k+h}$  such that  $\psi^\alpha(\bar{x} - f) = \alpha \bar{s}$  and  $\sum_{j=1}^{k+h} r^j \bar{s}_j = \bar{x} - f$ ,  $s_j \geq 0$  for  $j = 1, \dots, k$ , then there exists a positive integer  $\lambda$  such that  $\tilde{s} = \bar{s} + \lambda N \sum_{j=k+1}^{k+h} e^j \in R_f(r^1, \dots, r^{k+h})$ . Since  $\alpha_j = 0$ ,  $j = k + 1, \dots, k + h$ , then  $\psi^\alpha(\bar{x} - f) = \alpha \bar{s} = \alpha \tilde{s} \geq 1$ .

The above fact also implies that the linear program (13) admits a finite optimum for every  $r \in \mathbb{R}^q$ .

We show that  $\sum_{r \in \mathbb{R}^q} \psi^\alpha(r) s_r \geq 1$  is a valid inequality for  $R_f(\mathbb{R}^q)$ . The function  $\psi^\alpha$  is sublinear (the proof is similar to the one of Lemma 3.1). Therefore, for every  $s \in R_f(\mathbb{R}^q)$ , given  $\bar{x} = f + \sum_{r \in \mathbb{R}^q} r s_r$ , it follows that

$$\sum_{r \in \mathbb{R}^q} \psi^\alpha(r) s_r \geq \psi^\alpha(\bar{x} - f) \geq 1.$$

We may define  $\psi^\beta$  similarly. It now follows that the sets  $B_{\psi^\alpha}$  and  $B_{\psi^\beta}$  are lattice-free convex sets with  $f$  in their interior.

By definition,  $\psi^\alpha(r^j) \leq \alpha_j$  and  $\psi^\beta(r^j) \leq \beta_j$ ,  $j = 1, \dots, k + h$ .

Let  $\psi = \frac{1}{2}(\psi^\alpha + \psi^\beta)$ . We will show that  $\psi = \psi_B$ . Indeed,  $\psi(r^j) \leq \frac{1}{2}(\alpha_j + \beta_j) \leq \psi_B(r^j)$ ,  $j = 1, \dots, k + h$ . Thus  $\psi(r^j) \leq 1$  for  $j = 1, \dots, k$  and  $\psi(r^j) \leq 0$  for  $j = k + 1, \dots, k + h$ . In particular,  $f + r^1, \dots, f + r^k \in B_\psi$  and  $r^{k+1}, \dots, r^{k+h} \in \text{rec}(B_\psi)$ . Thus  $B_\psi \supseteq B$ . Since  $\psi$  is a convex combination of  $\psi^\alpha$  and  $\psi^\beta$ , it follows that  $\sum_{r \in \mathbb{R}^q} \psi(r) s_r \geq 1$  is a valid inequality for  $R_f(\mathbb{R}^q)$ . Thus  $B_\psi$  is a lattice-free convex set. Since  $B$  is maximal, it follows that  $B_\psi = B$ . Hence  $\psi = \psi_B$ .

Since  $\psi_B$  is extreme, it follows that  $\psi^\alpha = \psi^\beta = \psi_B$ . Hence  $\alpha_j = \beta_j = \psi_B(r^j)$ ,  $j = 1, \dots, k + h$ .  $\square$

**Acknowledgments.** This research was supported by NSF grant CMMI0653419, ONR grant N00014-09-1-0133, Progetto di Eccellenza 2008-2009 of the Fondazione Cassa di Risparmio di Padova e Rovigo, and a Mellon Fellowship.

**References**

[1] K. Andersen, Q. Louveaux, R. Weismantel, L. A. Wolsey, *Cutting Planes from Two Rows of a Simplex Tableau*, Proceedings of IPCO XII, Ithaca, New York (June 2007), Lecture Notes in Computer Science **4513**, 1–15.

- [2] K. Andersen, Q. Louveaux, R. Weismantel, *An Analysis of Mixed Integer Linear Sets Based on Lattice Point Free Convex Sets*, *Mathematics of Operations Research* **35** (2010), 233–256.
- [3] K. Andersen, Q. Louveaux, and R. Weismantel. *Certificates of Linear Mixed Integer Infeasibility*, *Operations Research Letters* **36** (2008), 734–738.
- [4] K. Andersen, Q. Louveaux and R. Weismantel, *Mixed-Integer Sets Associated from Two Rows of Two Adjacent Simplex Bases*, *Mathematical Programming B* **124** (2010) 455–480.
- [5] K. Andersen, C. Wagner, R. Weismantel, *On an Analysis of the Strength of Mixed Integer Cutting Planes from Multiple Simplex Tableau Rows*, *SIAM Journal on Optimization* **20** (2009) 967–982.
- [6] E. Balas, *Intersection Cuts – A New Type of Cutting Planes for Integer Programming*, *Operations Research* **19** (1971), 19–39.
- [7] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics **54**, American Mathematical Society, Providence, Rhode Island, 2002.
- [8] A. Basu, P. Bonami, G. Cornuéjols, F. Margot, *On the Relative Strength of Split, Triangle and Quadrilateral Cuts*, to appear in *Mathematical Programming*. DOI: 10.1007/s10107-009-0281-x
- [9] A. Basu, G. Cornuéjols, G. Zambelli, *Convex Sets and Minimal Sublinear Functions*, *Journal of Convex Analysis* **18**(2) (2011) (electronic version available at <http://www.heldermann.de>).
- [10] V. Borozan, G. Cornuéjols, *Minimal Valid Inequalities for Integer Constraints*, *Mathematics of Operations Research* **34** (2009), 538–546.
- [11] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Grundlehren der mathematischen Wissenschaften **99**, Berlin- Gttingen-Heidelberg, Springer, 1959.
- [12] J. B. Conway, *A Course in Functional Analysis*, Graduate Texts in Mathematics, Springer, Berlin, 1990.
- [13] G. Cornuéjols, F. Margot, *On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints*, *Mathematical Programming A* **120** (2009), 429–456.
- [14] S.S. Dey, L.A. Wolsey, *Lifting Integer Variables in Minimal Inequalities Corresponding to Lattice-Free Triangles*, IPCO 2008, Bertinoro, Italy (May 2008), *Lecture Notes in Computer Science* **5035**, 463–475.
- [15] S.S. Dey, L.A. Wolsey, *Constrained Infinite Group Relaxations of MIPs*, manuscript (March 2009), to appear in *SIAM Journal on Optimization*.
- [16] J. P. Doignon, *Convexity in crystallographic lattices*, *Journal of Geometry* **3** (1973), 71–85.
- [17] D. Espinoza, *Computing with Multi-Row Gomory Cuts*, Proceedings of IPCO XIII, Bertinoro, Italy, 2008.
- [18] R.G. Gomory, *Some Polyhedra Related to Combinatorial Problems*, *Linear Algebra and Applications* **2** (1969), 451–558.
- [19] R.E. Gomory, *Thoughts about Integer Programming*, 50th Anniversary Symposium of OR, University of Montreal, January 2007, and *Corner Polyhedra and Two-Equation Cutting Planes*, George Nemhauser Symposium, Atlanta, July 2007.
- [20] R. E. Gomory, E. L. Johnson, *Some Continuous Functions Related to Corner Polyhedra*, *Mathematical Programming* **3** (1972), 23–85.
- [21] J-B. Hiriart-Urruty, C. Lemaréchal, *Fundamentals of Convex Analysis*, Springer, Berlin, 2001.
- [22] A.H. Lenstra Jr., *Integer Programming with a Fixed Number of Variables*, *Mathematics of Operations Research* **8** (1983), 538–548.
- [23] L. Lovász, *Geometry of Numbers and Integer Programming*, *Mathematical Programming: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer, 1989, 177–210.
- [24] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.
- [25] G. Zambelli, *On Degenerate Multi-Row Gomory Cuts*, *Operations Research Letters* **37** (2009), 21–22.