

Minimal inequalities for an infinite relaxation of integer programs

Amitabh Basu
Carnegie Mellon University, abasu1@andrew.cmu.edu

Michele Conforti
Università di Padova, conforti@math.unipd.it

G rard Cornu jols *
Carnegie Mellon University and Universit  d'Aix-Marseille
gc0v@andrew.cmu.edu

Giacomo Zambelli
Universit  di Padova, giacomo@math.unipd.it

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Abstract

We show that maximal S -free convex sets are polyhedra when S is the set of integral points in some rational polyhedron of \mathbb{R}^n . This result extends a theorem of Lov sz characterizing maximal lattice-free convex sets. Our theorem has implications in integer programming. In particular, we show that maximal S -free convex sets are in one-to-one correspondence with minimal inequalities.

1 Introduction

Consider a mixed integer linear program, and the optimal tableau of the linear programming relaxation. We select n rows of the tableau, relative to n basic integer variables x_1, \dots, x_n . Let s_1, \dots, s_m denote the nonbasic variables. Let $f_i \geq 0$ be the value of x_i in the basic solution associated with the tableau, $i = 1, \dots, n$, and suppose $f \notin \mathbb{Z}^n$. The tableau restricted to these n rows is of the form

$$x = f + \sum_{j=1}^m r^j s_j, \quad x \geq 0 \text{ integral}, s \geq 0, \text{ and } s_j \in \mathbb{Z}, j \in I, \quad (1)$$

where $r^j \in \mathbb{R}^n$, $j = 1, \dots, m$, and I denotes the set of integer nonbasic variables.

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An important question in integer programming is to derive valid inequalities for (1), cutting off the current infeasible solution $x = f$, $s = 0$. We will consider a simplified model where the integrality conditions are relaxed on all nonbasic variables. On the other hand, we can present our results in a more general context, where the constraints $x \geq 0$, $x \in \mathbb{Z}^n$, are replaced by constraints $x \in S$, where S is the set of integral points in some given rational polyhedron such that $\dim(S) = n$, i.e. S contains $n + 1$ affinely independent points. Recall that a polyhedron $Ax \leq b$ is *rational* if the matrix A and vector b have rational entries.

So we study the following model, introduced by Johnson [8].

$$x = f + \sum_{j=1}^m r^j s_j, \quad x \in S, s \geq 0, \quad (2)$$

where $f \in \text{conv}(S) \setminus \mathbb{Z}^n$. Note that every inequality cutting off the point $(f, 0)$ can be expressed in terms of the nonbasic variables s only, and can therefore be written in the form $\sum_{j=1}^m \alpha_j s_j \geq 1$.

In this paper we are interested in “formulas” for deriving such inequalities. More formally, we are interested in functions $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the inequality

$$\sum_{j=1}^m \psi(r^j) s_j \geq 1$$

is valid for (2) for every choice of m and vectors $r^1, \dots, r^m \in \mathbb{R}^n$. We refer to such functions ψ as *valid functions* (with respect to f and S). Note that, if ψ is a valid function and ψ' is a function such that $\psi \leq \psi'$, then ψ' is also valid, and the inequality $\sum_{j=1}^m \psi'(r^j) s_j \geq 1$ is implied by $\sum_{j=1}^m \psi(r^j) s_j \geq 1$. Therefore we only need to investigate (pointwise) minimal valid functions.

Andersen, Louveaux, Weismantel, Wolsey [1] characterize minimal valid functions for the case $n = 2$, $S = \mathbb{Z}^2$. Borozan and Cornuéjols [6] extend this result to $S = \mathbb{Z}^n$ for any n . These papers and a result of Zambelli [11] show a one-to-one correspondence between minimal valid functions and maximal lattice-free convex sets with f in the interior. These results have been further generalized in [4]. Minimal valid functions for the case $S = \mathbb{Z}^n$ are intersection cuts [2].

Our interest in model (2) arose from a recent paper of Dey and Wolsey [7]. They introduce the notion of *S-free convex set* as a convex set without points of S in its interior, and show the connection between valid functions and *S-free convex sets* with f in their interior.

A class of valid functions can be defined as follows. A function ψ is *positively homogeneous* if $\psi(\lambda r) = \lambda \psi(r)$ for every $r \in \mathbb{R}^n$ and every $\lambda \geq 0$, and it is *subadditive* if $\psi(r) + \psi(r') \geq \psi(r + r')$ for all $r, r' \in \mathbb{R}^n$. A function ψ is *sublinear* if it is positively homogeneous and subadditive. It is easy to observe that sublinear functions are also convex.

Assume that ψ is a sublinear function such that the set

$$B_\psi = \{x \in \mathbb{R}^n \mid \psi(x - f) \leq 1\} \quad (3)$$

is *S-free*. Note that B_ψ is closed and convex because ψ is convex. Since ψ is positively homogeneous, $\psi(0) = 0$, thus f is in the interior of B_ψ . We claim that ψ is a valid function.

Indeed, given any solution (\bar{x}, \bar{s}) to (2), we have

$$\sum_{j=1}^m \psi(r^j) \bar{s}_j \geq \psi\left(\sum_{j=1}^m r^j \bar{s}_j\right) = \psi(\bar{x} - f) \geq 1,$$

where the first inequality follows from sublinearity and the last one follows from the fact that \bar{x} is not in the interior of B_ψ .

Dey and Wolsey [7] show that every minimal valid function ψ is sublinear and B_ψ is an S -free convex set with f in its interior. In this paper, we prove that if ψ is a minimal valid function, then B_ψ is a *maximal S -free convex set*.

In Section 2, we show that maximal S -free convex sets are polyhedra. Therefore a maximal S -free convex set $B \subseteq \mathbb{R}^n$ containing f in its interior can be uniquely written in the form $B = \{x \in \mathbb{R}^n : a_i(x - f) \leq 1, i = 1, \dots, k\}$. Let $\psi_B : \mathbb{R}^n \rightarrow \mathbb{R}$ be the function defined by

$$\psi_B(r) = \max_{i=1, \dots, k} a_i r, \quad \forall r \in \mathbb{R}^n. \quad (4)$$

It is easy to observe that the above function is sublinear and $B = \{x \in \mathbb{R}^n \mid \psi_B(x - f) \leq 1\}$. In Section 3 we will prove that every minimal valid function is of the form ψ_B for some maximal S -free convex set B containing f in its interior. Conversely, if B is a maximal S -free convex set containing f in its interior, then ψ_B is a minimal valid function.

2 Maximal S -free convex sets

Let $S \subseteq \mathbb{Z}^n$ be the set of integral points in some rational polyhedron of \mathbb{R}^n . We say that $B \subset \mathbb{R}^n$ is an *S -free convex set* if B is convex and does not contain any point of S in its interior. We say that B is a *maximal S -free convex set* if it is an S -free convex set and it is not properly contained in any S -free convex set. It follows from Zorn's lemma that every S -free convex set is contained in a maximal S -free convex set.

When $S = \mathbb{Z}^n$, an S -free convex set is called a *lattice-free convex set*. The following theorem of Lovász characterizes maximal lattice-free convex sets. A linear subspace or cone in \mathbb{R}^n is *rational* if it can be generated by rational vectors, i.e. vectors with rational coordinates.

Theorem 1. (Lovász [9]) *A set $B \subset \mathbb{R}^n$ is a maximal lattice-free convex set if and only if one of the following holds:*

- (i) *B is a polyhedron of the form $B = P + L$ where P is a polytope, L is a rational linear space, $\dim(B) = \dim(P) + \dim(L) = n$, B does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of B ;*
- (ii) *B is a hyperplane of \mathbb{R}^n that is not rational.*

Lovász only gives a sketch of the proof. A complete proof can be found in [4]. The next theorem is an extension of Lovász' theorem to maximal S -free convex sets.

Given a convex set $K \subset \mathbb{R}^n$, we denote by $\text{rec}(K)$ its recession cone and by $\text{lin}(K)$ its lineality space. Given a set $X \subseteq \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by X . Given a k -dimensional linear space V and a subset Λ of V , we say that Λ is a *lattice of V* if there exists a linear bijection $f : \mathbb{R}^k \rightarrow V$ such that $\Lambda = f(\mathbb{Z}^k)$.

Theorem 2. Let S be the set of integral points in some rational polyhedron of \mathbb{R}^n such that $\dim(S) = n$. A set $B \subset \mathbb{R}^n$ is a maximal S -free convex set if and only if one of the following holds:

- (i) B is a polyhedron such that $B \cap \text{conv}(S)$ has nonempty interior, B does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets.
- (ii) B is a half-space of \mathbb{R}^n such that $B \cap \text{conv}(S)$ has empty interior and the boundary of B is a supporting hyperplane of $\text{conv}(S)$.
- (iii) B is a hyperplane of \mathbb{R}^n such that $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ is not rational.

Furthermore, if (i) holds, the recession cone of $B \cap \text{conv}(S)$ is rational and it is contained in the lineality space of B .

We illustrate case (i) of the theorem in the plane in Figure 2. The question of the polyhedrality of maximal S -free convex sets was raised by Dey and Wolsey [7]. They proved that this is the case for a maximal S -free convex set B , under the assumptions that $B \cap \text{conv}(S)$ has nonempty interior and that the recession cone of $B \cap \text{conv}(S)$ is finitely generated and rational. Theorem 2 settles the question in general.

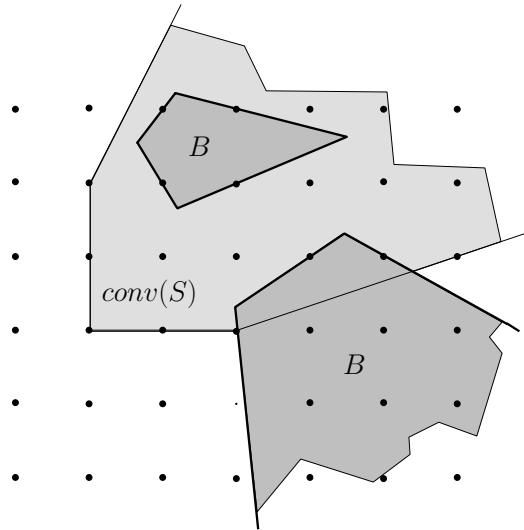


Figure 1: Two examples of S -free sets in the plane (case (i) of Theorem 2). The light gray region indicates $\text{conv}(S)$ and the dark gray regions illustrate the S -free sets. A jagged line indicates that the region extends to infinity.

To prove Theorem 2 we will need the following lemmas. The first one is proved in [4] and is an easy consequence of Dirichlet's theorem.

Lemma 3. Let $y \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$. For every $\varepsilon > 0$ and $\bar{\lambda} \geq 0$, there exists an integral point at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$.

Lemma 4. *Let B be an S -free convex set such that $B \cap \text{conv}(S)$ has nonempty interior. For every $r \in \text{rec}(B) \cap \text{rec}(\text{conv}(S))$, $B + \langle r \rangle$ is S -free.*

Proof. Let $C = \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ and $r \in C \setminus \{0\}$. Suppose by contradiction that there exists $y \in S \cap \text{int}(B + \langle r \rangle)$. We show that $y \in \text{int}(B) + \langle r \rangle$. If not, $(y + \langle r \rangle) \cap \text{int}(B) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $B + \langle r \rangle$, a contradiction. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda}r \in \text{int}(B)$, i.e. there exists $\varepsilon > 0$ such that B contains the open ball $B_\varepsilon(\bar{y})$ of radius ε centered at \bar{y} . Since $r \in C \subseteq \text{rec}(B)$, it follows that $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset B$. Since $y \in \mathbb{Z}^n$, by Lemma 3 there exists $z \in \mathbb{Z}^n$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \text{int}(B)$. Note that the half-line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ is in $\text{conv}(S)$, since $y \in S$ and $r \in \text{rec}(\text{conv}(S))$. Since $\text{conv}(S)$ is a rational polyhedron, for $\varepsilon > 0$ sufficiently small every integral point at distance at most ε from $\text{conv}(S)$ is in $\text{conv}(S)$. Therefore $z \in S$, a contradiction. \square

Proof of Theorem 2. The proof of the “if” part is standard, and it is similar to the proof for the lattice-free case (see [4]). We show the “only if” part. Let B be a maximal S -free convex set. If $\dim(B) < n$, then B is contained in some affine hyperplane K . Since K has empty interior, K is S -free, thus $B = K$ by maximality of B . Next we show that $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ is not rational. Suppose not. Then the linear subspace $L = \langle \text{lin}(B) \cap \text{rec}(\text{conv}(S)) \rangle$ is rational. Therefore the projection Λ of \mathbb{Z}^n onto L^\perp is a lattice of L^\perp (see, for example, Barvinok [3] p 284 problem 3). The projection S' of S onto L^\perp is a subset of Λ . Let B' be the projection of B onto L^\perp . Then $B' \cap \text{conv}(S')$ is the projection of $B \cap \text{conv}(S)$ onto L^\perp . Since B is a hyperplane, $\text{lin}(B) = \text{rec}(B)$. This implies that $B' \cap \text{conv}(S')$ is bounded : otherwise there is an unbounded direction $d \in L^\perp$ in $\text{rec}(B') \cap \text{rec}(\text{conv}(S'))$ and so $d + l \in \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ for some $l \in L$. Since $\text{rec}(B) \cap \text{rec}(\text{conv}(S)) = \text{lin}(B) \cap \text{rec}(\text{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Fix $\delta > 0$. Since Λ is a lattice and $S' \subseteq \Lambda$, there is a finite number of points at distance less than δ from the bounded set $B' \cap \text{conv}(S')$ in L^\perp . It follows that there exists $\varepsilon > 0$ such that every point of S' has distance at least ε from $B' \cap \text{conv}(S')$. Let $B'' = \{v + w \mid v \in B, w \in L^\perp, \|w\| \leq \varepsilon\}$. The set B'' is S -free by the choice of ε , but B'' strictly contains B , contradicting the maximality of B . Therefore (iii) holds when $\dim(B) < n$. Hence we may assume $\dim(B) = n$. If $B \cap \text{conv}(S)$ has empty interior, then there exists a hyperplane separating B and $\text{conv}(S)$ which is supporting for $\text{conv}(S)$. By maximality of B case (ii) follows.

Therefore we may assume that $B \cap \text{conv}(S)$ has nonempty interior. We show that B satisfies (i).

Claim 1. *There exists a rational polyhedron P such that:*

- i) $\text{conv}(S) \subset \text{int}(P)$,
- ii) *The set $K = B \cap P$ is lattice-free,*
- iii) *For every facet F of P , $F \cap K$ is a facet of K ,*
- iv) *For every facet F of P , $F \cap K$ contains an integral point in its relative interior.*

Since $\text{conv}(S)$ is a rational polyhedron, there exist integral A and b such that $\text{conv}(S) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$. The set $P' = \{x \in \mathbb{R}^n \mid Ax \leq b + \frac{1}{2}\mathbf{1}\}$ satisfies i). The set $B \cap P'$ is lattice-free since B is S -free and P' does not contain any point in $\mathbb{Z}^n \setminus S$, thus P' also satisfies ii). Let $\bar{A}x \leq \bar{b}$ be the system containing all inequalities of $Ax \leq b + \frac{1}{2}\mathbf{1}$ that define facets

of $B \cap P'$. Let $P_0 = \{x \in \mathbb{R}^n \mid \bar{A}x \leq \bar{b}\}$. Then P_0 satisfies *i*), *ii*), *iii*). See Figure 2 for an illustration.

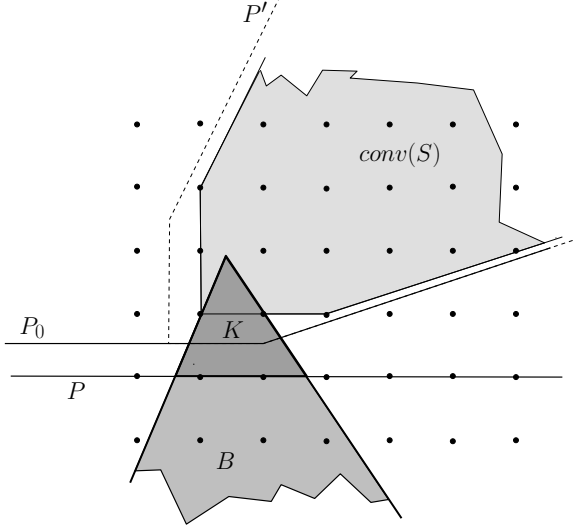


Figure 2: Illustration for Claim 1.

It will be more convenient to write P_0 as intersection of the half-spaces defining the facets of P_0 , $P_0 = \bigcap_{H \in \mathcal{F}_0} H$. We construct a sequence of rational polyhedra $P_0 \subset P_1 \subset \dots \subset P_t$ such that P_i satisfies *i*), *ii*), *iii*), $i = 1, \dots, t$, and such that P_t satisfies *iv*). Given P_i , we construct P_{i+1} as follows. Let $P_i = \bigcap_{H \in \mathcal{F}_i} H$, where \mathcal{F}_i is the set of half spaces defining facets of P_i . Let \bar{H} be a half-space in \mathcal{F}_i defining a facet of $B \cap P_i$ that does not contain an integral point in its relative interior; if no such \bar{H} exists, then P_i satisfies *iv*) and we are done. If $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ does not contain any integral point in its interior, let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\}$. Otherwise, since P_i is rational, among all integral points in the interior of $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ there exists one, say \bar{x} , at minimum distance from \bar{H} . Let H' be the half-space containing \bar{H} with \bar{x} on its boundary. Let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\} \cup \{H'\}$. Observe that H' defines a facet of P_{i+1} since \bar{x} is in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i+1} \setminus \{H'\}} H$ and it is on the boundary of H' . So *i*), *ii*), *iii*) are satisfied and P_{i+1} has fewer facets that violate *iv*) than P_i . \diamond

Let T be a maximal lattice-free convex set containing the set K defined in Claim 1. As remarked earlier, such a set T exists. By Theorem 1, T is a polyhedron with an integral point in the relative interior of each of its facets. Let H be a hyperplane that defines a facet of P . Since $K \cap H$ is a facet of K with an integral point in its relative interior, it follows that H defines a facet of T . This implies that $T \subset P$. Therefore we can write T as

$$T = P \cap \bigcap_{i=1}^k H_i, \quad (5)$$

where H_i are halfspaces. Let $\bar{H}_i = \mathbb{R}^n \setminus \text{int}(H_i)$, $i = 1, \dots, k$.

Claim 2. B is a polyhedron.

We first show that, for $i = 1, \dots, k$, $\mathbf{int}(B) \cap (\bar{H}_i \cap \text{conv}(S)) = \emptyset$. Consider $y \in \mathbf{int}(B) \cap \bar{H}_i$. Since $y \in \bar{H}_i$ and K is contained in T , $y \notin \mathbf{int}(K)$. Since $K = B \cap P$ and $y \in \mathbf{int}(B) \setminus \mathbf{int}(K)$, it follows that $y \notin \mathbf{int}(P)$. Hence $y \notin \text{conv}(S)$ because $\text{conv}(S) \subseteq \mathbf{int}(P)$.

Thus, for $i = 1, \dots, k$, there exists a hyperplane separating B and $\bar{H}_i \cap \text{conv}(S)$. Hence there exists a halfspace K_i such that $B \subset K_i$ and $\bar{H}_i \cap \text{conv}(S)$ is disjoint from the interior of K_i . We claim that the set $B' = \bigcap_{i=1}^k K_i$ is S -free. Indeed, let $y \in S$. Then y is not interior of T . Since $y \in \text{conv}(S)$ and $\text{conv}(S) \subseteq \mathbf{int}(P)$, y is in the interior of P . Hence, by (5), there exists $i \in \{1, \dots, k\}$ such that y is not in the interior of H_i . Thus $y \in \bar{H}_i \cap \text{conv}(S)$. By construction, y is not in the interior of K_i , hence y is not in the interior of B' . Thus B' is an S -free convex set containing B . Since B is maximal, $B' = B$. \diamond

Claim 3. $\text{lin}(K) = \text{rec}(K)$.

Let $r \in \text{rec}(K)$. We show $-r \in \text{rec}(K)$. By Lemma 4 applied to \mathbb{Z}^n , $K + \langle r \rangle$ is lattice-free. We observe that $B + \langle r \rangle$ is S -free. If not, let $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $S \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P + \langle r \rangle)$, hence $y \in \mathbf{int}(K + \langle r \rangle)$, a contradiction. Hence, by maximality of B , $B = B + \langle r \rangle$. Thus $-r \in \text{rec}(B)$. Suppose that $-r \notin \text{rec}(P)$. Then there exists a facet F of P that is not parallel to r . By construction, $F \cap K$ is a facet of K containing an integral point \bar{x} in its relative interior. The point \bar{x} is then in the interior of $K + \langle r \rangle$, a contradiction. \diamond

Claim 4. $\text{lin}(K)$ is rational.

Consider the maximal lattice-free convex set T containing K considered earlier. By Theorem 1, $\text{lin}(T) = \text{rec}(T)$, and $\text{lin}(T)$ is rational. Clearly $\text{lin}(T) \supseteq \text{lin}(K)$. Hence, if the claim does not hold, there exists a rational vector $r \in \text{lin}(T) \setminus \text{lin}(K)$. By (5), $r \in \text{lin}(P)$. Since $K = B \cap P$, $r \notin \text{lin}(B)$. Hence $B \subset B + \langle r \rangle$. We will show that $B + \langle r \rangle$ is S -free, contradicting the maximality of B . Suppose there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $\text{conv}(S) \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P) \subseteq \mathbf{int}(P) + \langle r \rangle$. Therefore $y \in \mathbf{int}(B \cap P) + \langle r \rangle$. Since $B \cap P \subseteq T$, then $y \in \mathbf{int}(T) + \langle r \rangle = \mathbf{int}(T)$ where the last equality follows from $r \in \text{lin}(T)$. This contradicts the fact that T is lattice-free. \diamond

By Lemma 4 and by the maximality of B , $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{rec}(B) \cap \text{rec}(\text{conv}(S))$.

Claim 5. $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ is rational.

Since $\text{lin}(K)$ and $\text{rec}(\text{conv}(S))$ are both rational, we only need to show $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{lin}(K) \cap \text{rec}(\text{conv}(S))$. The “ \supseteq ” direction follows from $B \supseteq K$. For the other direction, note that, since $\text{conv}(S) \subseteq P$, we have $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) \subseteq \text{lin}(B) \cap \text{rec}(P) = \text{lin}(B \cap P) = \text{lin}(K)$, hence $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) \subseteq \text{lin}(K) \cap \text{rec}(\text{conv}(S))$. \diamond

Claim 6. Every facet of B contains a point of S in its relative interior.

Let L be the linear space generated by $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$. By Claim 5, L is rational. Let B', S', Λ be the projections of B, S, \mathbb{Z}^n , respectively, onto L^\perp . Since L is rational, Λ is a lattice of L^\perp and $S' = \text{conv}(S') \cap \Lambda$. Also, B' is a maximal S' -free convex set of L^\perp , since for any S' -free set D of L^\perp , $D + L$ is S -free. Note that $\text{lin}(B) \cap \text{rec}(\text{conv}(S)) = \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ implies that $B' \cap \text{conv}(S')$ is bounded. Otherwise there is an unbounded direction $d \in L^\perp$ in $\text{rec}(B') \cap \text{rec}(\text{conv}(S'))$ and so $d + l \in \text{rec}(B) \cap \text{rec}(\text{conv}(S))$ for some

$l \in L$. Since $\text{rec}(B) \cap \text{rec}(\text{conv}(S)) = \text{lin}(B) \cap \text{rec}(\text{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Let $B' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t\}$. Given $\varepsilon > 0$, let $\bar{B} = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \beta_t + \varepsilon\}$. The polyhedron $\text{conv}(S') \cap \bar{B}$ is a polytope since it has the same recession cone as $\text{conv}(S') \cap B'$. The polytope $\text{conv}(S') \cap \bar{B}$ contains points of S' in its interior by the maximality of B' . Since Λ is a lattice of L^\perp , $\text{int}(\text{conv}(S') \cap \bar{B})$ has a finite number of points in S' , hence there exists one minimizing $\alpha_t x$, say z . By construction, the polyhedron $B'' = \{x \in L^\perp \mid \alpha_i x \leq \beta_i, i = 1, \dots, t-1, \alpha_t x \leq \alpha_t z\}$ does not contain any point of S' in its interior and contains B' . By the maximality of B' , $B' = B''$ hence B' contains z in its relative interior, and B contains a point of S in its relative interior. \square

Corollary 5. *For every maximal S -free convex set B there exists a maximal lattice-free convex set K such that, for every facet F of B , $F \cap K$ is a facet of K .*

Proof. Let K be defined as in Claim 1 in the proof of Theorem 2. It follows from the proof that K is a maximal lattice-free convex set with the desired properties. \square

3 Minimal valid functions

In this section we study minimal valid functions. We find it convenient to state our results in terms of an infinite model introduced by Dey and Wolsey [7].

Throughout this section, $S \subseteq \mathbb{Z}^n$ is a set of integral points in some rational polyhedron of \mathbb{R}^n such that $\dim(S) = n$, and f is a point in $\text{conv}(S) \setminus \mathbb{Z}^n$. Let $R_{f,S}$ be the set of all infinite dimensional vectors $s = (s_r)_{r \in \mathbb{R}^n}$ such that

$$\begin{aligned} f + \sum_{r \in \mathbb{R}^n} r s_r &\in S \\ s_r &\geq 0, \quad r \in \mathbb{R}^n \\ s &\text{ has finite support} \end{aligned} \tag{6}$$

where s has *finite support* means that s_r is zero for all but a finite number of $r \in \mathbb{R}^n$.

A function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is *valid* (with respect to f and S) if the linear inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1 \tag{7}$$

is satisfied by every $s \in R_{f,S}$. Note that this definition coincides with the one we gave in the introduction.

Given two functions ψ, ψ' we say that ψ' *dominates* ψ if $\psi'(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$. A valid function ψ is *minimal* if there is no valid function $\psi' \neq \psi$ that dominates ψ .

Theorem 6. *For every minimal valid function ψ , there exists a maximal S -free convex set B with f in its interior such that ψ_B dominates ψ . Furthermore, if B is a maximal S -free convex set containing f in its interior, then ψ_B is a minimal valid function.*

We will need the following lemma.

Lemma 7. *Every valid function is dominated by a sublinear valid function.*

Sketch of proof. Given a valid function ψ , define the following function $\bar{\psi}$. For all $\bar{r} \in \mathbb{R}^n$, let $\bar{\psi}(\bar{r}) = \inf\{\sum_{r \in \mathbb{R}^n} \psi(r)s_r \mid \sum_{r \in \mathbb{R}^n} rs_r = \bar{r}, s \geq 0 \text{ with finite support}\}$. Following the proof of Lemma 18 in [4] one can show that $\bar{\psi}$ is a valid sublinear function that dominates ψ . \square

Given a valid sublinear function ψ , the set $B_\psi = \{x \in \mathbb{R}^n \mid \psi(x-f) \leq 1\}$ is closed, convex, and contains f in its interior. Since ψ is a valid function, B_ψ is S -free. Indeed the interior of B_ψ is $\text{int}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) < 1\}$. Its boundary is $\text{bd}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) = 1\}$, and its recession cone is $\text{rec}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) \leq 0\}$.

Before proving Theorem 6, we need the following general theorem about sublinear functions. Let K be a closed, convex set in \mathbb{R}^n with the origin in its interior. The *polar* of K is the set $K^* = \{y \in \mathbb{R}^n \mid ry \leq 1 \text{ for all } r \in K\}$. Clearly K^* is closed and convex, and since $0 \in \text{int}(K)$, it is well known that K^* is bounded. In particular, K^* is a compact set. Also, since $0 \in K$, $K^{**} = K$. Let

$$\hat{K} = \{y \in K^* \mid \exists x \in K \text{ such that } xy = 1\}. \quad (8)$$

Note that \hat{K} is contained in the relative boundary of K^* . Let $\rho_K : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$\rho_K(r) = \sup_{y \in \hat{K}} ry, \quad \text{for all } r \in \mathbb{R}^n. \quad (9)$$

It is easy to show that ρ_K is sublinear.

Theorem 8 (Basu et al. [5]). *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. Then $K = \{r \in \mathbb{R}^n \mid \rho_K(r) \leq 1\}$. Furthermore, for every sublinear function σ such that $K = \{r \mid \sigma(r) \leq 1\}$, we have $\rho_K(r) \leq \sigma(r)$ for every $r \in \mathbb{R}^n$.*

Remark 9. *Let $K \subset \mathbb{R}^n$ be a polyhedron containing the origin in its interior. Let $a_1, \dots, a_t \in \mathbb{R}^n$ such that $K = \{r \in \mathbb{R}^n \mid a_i r \leq 1, i = 1, \dots, t\}$. Then $\rho_K(r) = \max_{i=1, \dots, t} a_i r$.*

Proof. The polar of K is $K^* = \text{conv}\{0, a_1, \dots, a_t\}$ (see Theorem 9.1 in Schrijver [10]). Furthermore, \hat{K} is the union of all the facets of K^* that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{i=1, \dots, t} a_i r$$

for all $r \in \mathbb{R}^n$. \square

Remark 10. *Let B be a closed S -free convex set in \mathbb{R}^n with f in its interior, and let $K = B - f$. Then ρ_K is a valid function.*

Proof: Let $s \in R_{f,S}$. Then $x = f + \sum_{r \in \mathbb{R}^n} rs_r$ is in S , therefore $x \notin \text{int}(B)$ because B is S -free. By Theorem 8, $\rho_K(x-f) \geq 1$. Thus

$$1 \leq \rho_K\left(\sum_{r \in \mathbb{R}^n} rs_r\right) \leq \sum_{r \in \mathbb{R}^n} \rho_K(r)s_r,$$

where the second inequality follows from the sublinearity of ρ_K . \square

Lemma 11. *Let C be a closed S -free convex set containing f in its interior, and let $K = C - f$. There exists a maximal S -free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1, i = 1, \dots, k\}$ such that $a_i \in \mathbf{cl}(\text{conv}(\hat{K}))$ for $i = 1, \dots, k$.*

Proof. Since C is an S -free convex set, it is contained in some maximal S -free convex set T . The set T satisfies one of the statements (i)-(iii) of Theorem 2. By assumption, $f \in \text{conv}(S)$ and f is in the interior of C . Since $\dim(S) = n$, $\text{conv}(S)$ is a full dimensional polyhedron, thus $\mathbf{int}(C \cap \text{conv}(S)) \neq \emptyset$. This implies that $\mathbf{int}(T \cap \text{conv}(S)) \neq \emptyset$, hence case (i) applies.

Thus T is a polyhedron and $\text{rec}(T \cap \text{conv}(S)) = \text{lin}(T) \cap \text{rec}(\text{conv}(S))$ is rational. Let us choose T such that the dimension of $\text{lin}(T)$ is largest possible.

Since T is a polyhedron containing f in its interior, there exists $D \in \mathbb{R}^{t \times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0, i = 1, \dots, t$, and $T = \{x \in \mathbb{R}^n \mid D(x - f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x \in C} d_i(x - f) = 1$ where d_i denotes the i th row of $D, i = 1, \dots, t$. By our assumption, $\sup_{r \in K} d_i r = 1$. Therefore $d_i \in K^*$, since $d_i r \leq 1$ for all $r \in K$. Furthermore $d_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r \in K} d_i r = 1$.

Let $P = \{x \in \mathbb{R}^n \mid D(x - f) \leq e\}$. Note that $\text{lin}(P) = \text{lin}(T)$. By our choice of $T, P + \langle r \rangle$ is not S -free for any $r \in \text{rec}(\text{conv}(S)) \setminus \text{lin}(P)$, otherwise P would be contained in a maximal S -free convex set whose lineality space contains $\text{lin}(T) + \langle r \rangle$, a contradiction.

Let $L = \langle \text{rec}(P \cap \text{conv}(S)) \rangle$. Since $\text{lin}(P) = \text{lin}(T)$, L is a rational space. Note that $L \subseteq \text{lin}(P)$, implying that $d_i \in L^\perp$ for $i = 1, \dots, t$.

We observe next that we may assume that $P \cap \text{conv}(S)$ is bounded. Indeed, let $\bar{P}, \bar{S}, \Lambda$ be the projections onto L^\perp of P, S , and \mathbb{Z}^n , respectively. Since L is a rational space, Λ is a lattice of L^\perp and $\bar{S} = \text{conv}(\bar{S}) \cap \Lambda$. Note that $\bar{P} \cap \text{conv}(\bar{S})$ is bounded, since $L \supseteq \text{rec}(P \cap \text{conv}(S))$. If we are given a maximal \bar{S} -free convex set \bar{B} in L^\perp such that $\bar{B} = \{x \in L^\perp \mid a_i(x - f) \leq 1, i = 1, \dots, h\}$ and $a_i \in \text{conv}\{d_1, \dots, d_t\}$ for $i = 1, \dots, h$, then $B = \bar{B} + L$ is the set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1, i = 1, \dots, h\}$. Since \bar{B} contains a point of \bar{S} in the relative interior of each of its facets, B contains a point of S in the relative interior of each of its facets, thus B is a maximal S -free convex set.

Thus we assume that $P \cap \text{conv}(S)$ is bounded, so $\dim(L) = 0$. If all facets of P contain a point of S in their relative interior, then P is a maximal S -free convex set, thus the statement of the lemma holds. Otherwise we describe a procedure that replaces one of the inequalities defining a facet of P without any point of S in its relative interior with an inequality which is a convex combination of the inequalities of $D(x - f) \leq e$, such that the new polyhedron thus obtained is S -free and has one fewer facet without points of S in its relative interior. More formally, suppose the facet of P defined by $d_1(x - f) \leq 1$ does not contain any point of S in its relative interior. Given $\lambda \in [0, 1]$, let

$$P(\lambda) = \{x \in \mathbb{R}^n \mid [\lambda d_1 + (1 - \lambda)d_2](x - f) \leq 1, \quad d_i(x - f) \leq 1 \quad i = 2, \dots, t\}.$$

Note that $P(1) = P$ and $P(0)$ is obtained from P by removing the inequality $d_1(x - f) \leq 1$. Furthermore, given $0 \leq \lambda' \leq \lambda'' \leq 1$, we have $P(\lambda') \supseteq P(\lambda'')$.

Let r_1, \dots, r_m be generators of $\text{rec}(\text{conv}(S))$. Note that, since $P \cap \text{conv}(S)$ is bounded, for every $j = 1, \dots, m$ there exists $i \in \{1, \dots, t\}$ such that $d_i r_j > 0$. Let r_1, \dots, r_h be the

generators of $\text{rec}(\text{conv}(S))$ satisfying

$$\begin{aligned} d_1 r_j &> 0 \\ d_i r_j &\leq 0 \quad i = 2, \dots, t. \end{aligned}$$

Note that, if no such generators exist, then $P(0) \cap \text{conv}(S)$ is bounded. Otherwise $P(\lambda) \cap \text{conv}(S)$ is bounded if and only if, for $j = 1, \dots, h$

$$[\lambda d_1 + (1 - \lambda) d_2] r_j > 0.$$

This is the case if and only if $\lambda > \lambda^*$, where

$$\lambda^* = \max_{j=1, \dots, h} \frac{-d_2 r_j}{(d_1 - d_2) r_j}.$$

Let r^* be one of the vectors r_1, \dots, r_h attaining the maximum in the previous equation. Then $r^* \in \text{rec}(P(\lambda^*) \cap \text{conv}(S))$.

Note that $P(\lambda^*)$ is not S -free otherwise $P(\lambda^*) + \langle r^* \rangle$ is S -free by Lemma 4, and so is $P + \langle r^* \rangle$, a contradiction.

Thus $P(\lambda^*)$ contains a point of S in its interior. That is, there exists a point $\bar{x} \in S$ such that $[\lambda^* d_1 + (1 - \lambda^*) d_2](\bar{x} - f) < 1$ and $d_i(\bar{x} - f) < 1$ for $i = 2, \dots, t$. Since P is S -free, $d_1(\bar{x} - f) > 1$. Thus there exists $\bar{\lambda} > \lambda^*$ such that $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](\bar{x} - f) = 1$. Note that, since $P(\bar{\lambda}) \cap \text{conv}(S)$ is bounded, there is a finite number of points of S in the interior of $P(\bar{\lambda})$. So we may choose \bar{x} such that $\bar{\lambda}$ is maximum. Thus $P(\bar{\lambda})$ is S -free and \bar{x} is in the relative interior of the facet of $P(\bar{\lambda})$ defined by $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](x - f) \leq 1$.

Note that, for $i = 2, \dots, t$, if $d_i(x - f) \leq 1$ defines a facet of P with a point of S in its relative interior, then it also defines a facet of $P(\bar{\lambda})$ with a point of S in its relative interior, because $P \subset P(\bar{\lambda})$. Thus repeating the above construction at most t times, we obtain a set B satisfying the lemma. \square

Remark 12. Let C and K be as in Lemma 11. Given any maximal S -free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1, i = 1, \dots, k\}$ containing C , then $a_1, \dots, a_k \in K^*$. If $\text{rec}(C)$ is not full dimensional, then the origin is not an extreme point of K^* . Since all extreme points of K^* are contained in $\{0\} \cup \hat{K}$, in this case $\text{cl}(\text{conv}(\hat{K})) = K^*$. Therefore, when $\text{rec}(C)$ is not full dimensional, every maximal S -free convex set containing C satisfies the statement of Lemma 11.

Proof of Theorem 6.

We first show that any valid function is dominated by a function of the form ψ_B , for some maximal S -free convex set B containing f in its interior.

Let ψ be a valid function. By Lemma 7, we may assume that ψ is sublinear. Let $K = \{r \in \mathbb{R}^n \mid \psi(r) \leq 1\}$, and let \hat{K} be defined as in (8). Note that $K = B_\psi - f$. Thus, by Remark 10, $\sum_{r \in \mathbb{R}^n} \rho_K(r) s_r \geq 1$ is valid for $R_{f,S}$. Since ψ is sublinear, it follows from Theorem 8 that $\rho_K(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$.

By Lemma 11, there exists a maximal S -free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1, i = 1, \dots, k\}$ such that $a_i \in \text{cl}(\text{conv}(\hat{K}))$ for $i = 1, \dots, k$.

Then

$$\psi(r) \geq \rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{y \in \text{cl}(\text{conv}(\hat{K}))} yr \geq \max_{i=1, \dots, k} a_i r = \psi_B(r).$$

This shows that ψ_B dominates ψ for all $r \in \mathbb{R}^n$.

To complete the proof of the theorem, we need to show that, given a maximal S -free convex set B , the function ψ_B is minimal. Consider any valid function ψ dominating ψ_B . Then $B_\psi \supseteq B$ and B_ψ is S -free. By maximality of B , $B = B_\psi$. By Theorem 8 and Remark 9, $\psi_B(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$, proving $\psi = \psi_B$. □

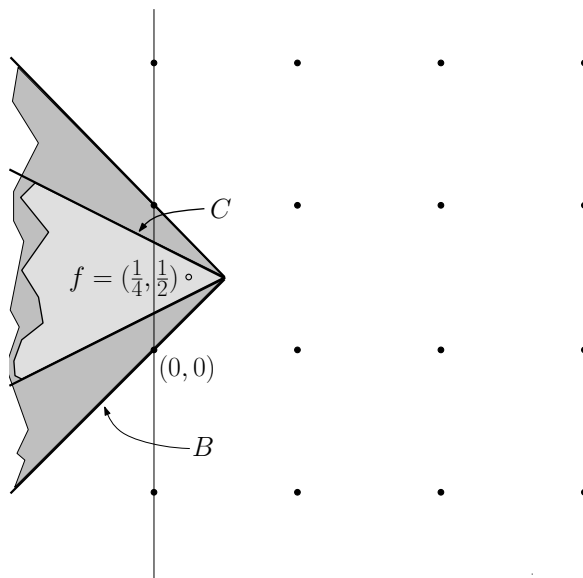


Figure 3: Illustration for Example 13

Example 13. We illustrate the ideas behind the proof in the following two-dimensional example. Consider $f = (\frac{1}{4}, \frac{1}{2})$ and $S = \{(x_1, x_2) \mid x_1 \geq 0\}$. See Figure 3. Then the function $\psi(r) = \max\{4r_1 + 8r_2, 4r_1 - 8r_2\}$ is a valid linear inequality for $R_{f,S}$. The corresponding B_ψ is $\{(x_1, x_2) \mid 4(x_1 - \frac{1}{4}) + 8(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 8(x_2 - \frac{1}{2}) \leq 1\}$. Note that B_ψ is not a maximal S -free convex set and it corresponds to C in Lemma 11. Following the procedure outlined in the proof, we obtain the maximal S -free convex set $B = \{(x_1, x_2) \mid 4(x_1 - \frac{1}{4}) + 4(x_2 - \frac{1}{2}) \leq 1, 4(x_1 - \frac{1}{4}) - 4(x_2 - \frac{1}{2}) \leq 1\}$. Then, $\psi_B(r) = \max\{4r_1 + 4r_2, 4r_1 - 4r_2\}$ and ψ_B dominates ψ .

Remark 14. Note that ψ is nonnegative if and only if $\text{rec}(B_\psi)$ is not full-dimensional. It follows from Remark 12 that, for every maximal S -free convex set B containing B_ψ , we have $\psi_B(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$ when ψ is nonnegative.

A statement similar to the one of Theorem 6 was shown by Borozan-Cornuéjols [6] for a model similar to (6) when $S = \mathbb{Z}^n$ and the vectors s are elements of $\mathbb{R}^{\mathbb{Q}^n}$. In this case, it is

easy to show that, for every valid inequality $\sum_{r \in \mathbb{Q}^n} \psi(r) s_r \geq 1$, the function $\psi : \mathbb{Q}^n \rightarrow \mathbb{R}$ is nonnegative. Remark 14 explains why in this context it is much easier to prove that minimal inequalities arise from maximal lattice-free convex sets.

References

- [1] K. Andersen, Q. Louveaux, R. Weismantel, L. A. Wolsey, Cutting Planes from Two Rows of a Simplex Tableau, *Proceedings of IPCO XII*, Ithaca, New York (June 2007), Lecture Notes in Computer Science 4513, 1-15.
- [2] E. Balas, Intersection cuts – A new type of cutting planes for integer programming, *Operations Research* **19** (1971) 19-39.
- [3] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, Rhode Island, 2002.
- [4] A. Basu, M. Conforti, G. Cornuéjols, G. Zambelli, Maximal lattice-free convex sets in linear subspaces, manuscript (March 2009).
- [5] A. Basu, G. Cornuéjols, G. Zambelli, Convex Sets and Minimal Sublinear Functions, manuscript (March 2009).
- [6] V. Borozan, G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, *Mathematics of Operations Research* **34** (2009) 538-546.
- [7] S.S. Dey, L.A. Wolsey, Constrained Infinite Group Relaxations of MIPs, manuscript (March 2009).
- [8] E.L. Johnson, Characterization of facets for multiple right-hand side choice linear programs, *Mathematical Programming Study* **14** (1981), 112-142.
- [9] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developements and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989), 177-210.
- [10] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York (1986).
- [11] G. Zambelli, On Degenerate Multi-Row Gomory Cuts, *Operations Research Letters* **37** (2009), 21-22.