Unique lifting of integer variables in minimal inequalities

Amitabh Basu[1], Manoel Campêlo[2,6], Michele Conforti[3,8],
Gérard Cornuéjols[4,7], Giacomo Zambelli[5,8]

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Abstract

This paper contributes to the theory of cutting planes for mixed integer linear programs (MILPs). Minimal valid inequalities are well understood for a relaxation of an MILP in tableau form where all the nonbasic variables are continuous; they are derived using the gauge function of maximal lattice-free convex sets. In this paper we study lifting functions for the nonbasic *integer* variables starting from such minimal valid inequalities. We characterize precisely when the lifted coefficient is equal to the coefficient of the corresponding continuous variable in every minimal lifting. The answer is a nonconvex region that can be obtained as a finite union of convex polyhedra. We also establish a necessary and sufficient condition for the uniqueness of the lifting function.

1 Introduction

In the context of mixed integer linear programming, there has been a renewed interest recently in the study of cutting planes that cut off a basic solution of the linear programming relaxation. More precisely, consider a mixed integer linear set in which the variables are partitioned into a basic set B and a nonbasic set N, and $K \subseteq B \cup N$ indexes the integer variables:

$$\begin{array}{rcll} x_i & = & f_i - \sum_{j\in N} a_{ij}x_j & \text{for } i \in B \\ x & \geq & 0 & \\ x_k & \in & \mathbb{Z} & \text{for } k \in K. \end{array} \tag{1}$$

Let X be the relaxation of (1) obtained by dropping the nonnegativity restriction on all the basic variables x_i, $i \in B$. The convex hull of X is the *corner polyhedron* introduced

[1]Department of Mathematics, University of California, Davis, CA 95616. abasu@math.ucdavis.edu
[2]Departamento de Estatística e Matemática Aplicada, Universidade Federal do Ceará, Brazil. mcampelo@lia.ufc.br
[3]Dipartimento di Matematica Pura e Applicata, Universitá di Padova, Via Trieste 63, 35121 Padova, Italy. conforti@math.unipd.it
[4]Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213. gc0v@andrew.cmu.edu
[5]London School of Economics and Political Sciences, Houghton Street, London WC2A 2AE. G.Zambelli@lse.ac.uk
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by Gomory [16] (see also [17]). Note that, for any $i \in B \setminus K$, the equation $x_i = f_i - \sum_{j \in N} a_{ij} x_j$ can be removed from the formulation of X since it just defines variable x_i. Therefore, throughout the paper, we will assume $B \subseteq K$, i.e. all basic variables are integer. Andersen, Louveaux, Weismantel and Wolsey [2] studied the corner polyhedron when $|B| = 2$ and $B = K$, i.e. all nonbasic variables are continuous. They give a complete characterization of the corner polyhedron using intersection cuts (Balas [3]) arising from splits, triangles and quadrilaterals. This very elegant result has been extended to $|B| > 2$ and $B = K$ by showing a correspondence between minimal valid inequalities and maximal lattice-free convex sets [8], [11]. These results and their extensions [9], [15] are best described in an infinite model, which we motivate next.

1.1 The Infinite Model

A classical family of cutting planes for (1) is that of Gomory mixed integer cuts. For a given row $i \in B$ of the tableau, the Gomory mixed integer cut is of the form $\sum_{j \in N \setminus K} \psi(a_{ij}) x_j + \sum_{j \in N \cap K} \pi(a_{ij}) x_j \geq 1$ where ψ and π are functions given by simple formulas. A nice feature of the Gomory mixed integer cut is that, for fixed f_i, the same functions ψ, π are used for any possible choice of the a_{ij}s in (1). It is well known that the Gomory mixed integer cuts are also valid for X. More generally, let a^j be the vector with entries a_{ij}, $i \in B$; we are interested in pairs (ψ, π) of functions such that the inequality $\sum_{j \in N \setminus K} \psi(a^j) x_j + \sum_{j \in N \cap K} \pi(a^j) x_j \geq 1$ is valid for X for any possible number of nonbasic variables and any choice of the nonbasic coefficients a_{ij}. Since we are interested in nonredundant inequalities, we can assume that the function (ψ, π) is pointwise minimal. While a general characterization of minimal valid functions seems hopeless (see for example [7]), when $N \cap K = \emptyset$ the minimal valid functions ψ are well understood in terms of maximal lattice-free convex sets, as already mentioned. Starting from such a minimal valid function ψ, an interesting question is how to generate a function π such that (ψ, π) is valid and minimal. Recent papers [13], [14] study when such a function π is unique. Here we prove two theorems that generalize and unify results from these two papers.

In order to formalize the concept of valid function (ψ, π), we introduce the following infinite model. In the setting below, we also allow further linear constraints on the basic variables. Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$ (for example, S could be the set of nonnegative integer points). Let $f \in \mathbb{R}^n \setminus S$. Consider the following infinite relaxation of (1), introduced in [15].

$$
\begin{aligned}
x = \ & f + \sum_{r \in \mathbb{R}^n} r s_r + \sum_{r \in \mathbb{R}^n} r y_r, \\
& x \in S, \\
& s_r \in \mathbb{R}_+, \ \forall r \in \mathbb{R}^n, \\
& y_r \in \mathbb{Z}_+, \ \forall r \in \mathbb{R}^n, \\
& s, y \text{ have finite support}
\end{aligned}
\tag{2}
$$

where the nonbasic continuous variables have been renamed s and the nonbasic integer variables have been renamed y, and where an infinite dimensional vector has *finite support* if it has a finite number of nonzero entries. Given two functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$, (ψ, π) is said to be *valid* for (2) if the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \sum_{r \in \mathbb{R}^n} \pi(r) y_r \geq 1$ holds for every (x, s, y)

satisfying (2). We also consider the infinite model where we only have continuous nonbasic variables.

$$\begin{aligned} x = \ & f + \sum_{r \in \mathbb{R}^n} r s_r \\ & x \in S, \\ & s_r \in \mathbb{R}_+, \ \forall r \in \mathbb{R}^n, \\ & s \text{ has finite support.} \end{aligned} \tag{3}$$

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be *valid* for (3) if the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ holds for every (x, s) satisfying (3). Given a valid function ψ for (3), a function π is a *lifting* of ψ if (ψ, π) is valid for (2). One is interested only in (pointwise) *minimal valid functions*, since non-minimal ones are implied by some minimal valid function. If ψ is a minimal valid function for (3) and π is a lifting of ψ such that (ψ, π) is a minimal valid function for (2) then we say that π is a *minimal lifting* of ψ. It can be shown, using Zorn's Lemma, that for every lifting π of ψ there exists some minimal lifting π' of ψ such that $\pi' \leq \pi$.

1.2 Sequence Independent Lifting and Unique Lifting Functions

While minimal valid functions for (3) have a simple characterization [9], minimal valid functions for (2) are not well understood. A general idea to derive minimal valid functions for (2) is to start from some minimal valid function ψ for (3), and construct a minimal lifting π of ψ. While there is no general technique to compute such a minimal lifting π, it is known that there exists a region R_ψ, containing the origin in its interior, where ψ coincides with π for any minimal lifting π. This latter fact was observed by Dey and Wolsey [14] for the case of $S = \mathbb{Z}^2$ and by Conforti, Cornuéjols and Zambelli [13] for the general case. In this paper we give a precise description of the region R_ψ (Theorem 4). The importance of this region comes from the fact that for any ray $r \in R_\psi$, the minimal lifting coefficient $\pi(r)$ is unique, i.e. it is the same for every minimal lifting. Thus, we get *sequence independent* lifting coefficients for the rays r in R_ψ. Moreover, these coefficients can be computed directly from the function ψ for which we have more direct tools. These ideas are related to the results of Balas and Jeroslow [4].

Furthermore, it is remarked in [13] that, if π is a minimal lifting of ψ, then $\pi(r) = \pi(r')$ for every $r, r' \in \mathbb{R}^n$ such that $r - r' \in \mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S))$. Therefore the coefficients of any minimal lifting π are uniquely determined in the region $R_\psi + (\mathbb{Z}^n \cap \operatorname{lin}(\operatorname{conv}(S)))$ (throughout this paper, we use $+$ to denote the Minkowski sum of two sets). In particular, whenever $\mathbb{R}^n$ can be covered by translating R_ψ by integer vectors in $\operatorname{lin}(\operatorname{conv}(S))$, the function ψ has a unique minimal lifting π. As mentioned above, if ψ has a unique minimal lifting, we can compute the best possible coefficients for all the integer variables in our problem, in a *sequence independent* manner. Thus, it is very useful to recognize the minimal valid functions ψ with unique minimal liftings. The second main result in this paper (Theorem 5) is to show that, for the case when $S = \mathbb{Z}^n$, the covering property is in fact a necessary and sufficient condition for the uniqueness of minimal liftings : if $R_\psi + \mathbb{Z}^n \neq \mathbb{R}^n$, then there are at least two distinct minimal liftings for ψ. Theorem 5 thus converts the question of recognizing minimal valid functions ψ that have a unique lifting function to the geometric question of covering $\mathbb{R}^n$

by lattice translates of the region R_ψ. This equivalence is utilized by Basu, Cornuéjols and Köppe to study the unique lifting properties of certain families of minimal valid functions [10].

2 Overview of the Main Results

Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$, and let $f \in \mathbb{R}^n \setminus S$. To state our main results, we need to explain the characterization of minimal valid functions for (3).

2.1 Minimal Valid Functions and Maximal Lattice-Free Convex Sets

We say that a convex set $B \subseteq \mathbb{R}^n$ is *S-free* if B does not contain any point of S in its interior. When $S = \mathbb{Z}^n$, S-free convex sets are called *lattice-free convex sets.* A set B is a *maximal S-free* convex set if it is an S-free convex set that is not properly contained in any S-free convex set. It was proved in [9] that maximal S-free convex sets are polyhedra containing a point of S in the relative interior of each facet. The following characterization of maximal S-free convex sets and the subsequent remark will be needed in the proofs, but the reader can skip them for now.

Theorem 1. *[9] A full-dimensional convex set B is a maximal S-free convex set if and only if B is a polyhedron that does not contain any point of S in its interior and each facet of B contains a point of S in its relative interior. Furthermore if $B \cap \operatorname{conv}(S)$ has nonempty interior, then $\operatorname{lin}(B)$ contains $\operatorname{rec}(B \cap \operatorname{conv}(S))$ implying that $\operatorname{rec}(B \cap \operatorname{conv}(S)) = \operatorname{lin}(B \cap \operatorname{conv}(S))$, and $\operatorname{lin}(B \cap \operatorname{conv}(S))$ is a rational subspace.*

Remark 2. *The proof of Theorem 1 in [9] implies the following. Given a maximal S-free convex set B, there exists $\delta > 0$ such that no point of $S \setminus B$ is at a distance less than δ from B.*

Given an S-free polyhedron $B \subseteq \mathbb{R}^n$ containing f in its interior, B can be uniquely written in the form

$$B = \{x \in \mathbb{R}^n : \; a_i(x - f) \le 1, \; i \in I\}, \tag{4}$$

where I is a finite set of indices in one-to-one correspondence with the facets of B.

Let $\psi_B : \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$\psi_B(r) = \max_{i \in I} a_i r, \quad \forall r \in \mathbb{R}^n. \tag{5}$$

Theorem 3. *[9] If B is a maximal S-free convex set containing f in its interior, then ψ_B is a minimal valid function for (3).*

Conversely, let ψ be a minimal valid function for (3). Then the set

$$B_\psi := \{x \in \mathbb{R}^n \,|\, \psi(x - f) \le 1\}$$

is a maximal S-free convex set containing f in its interior, and $\psi = \psi_{B_\psi}$.

It follows easily from the formula in (5) and Theorem 3 that minimal valid functions for (3) are *sublinear.* In particular, $\psi(r_1) + \psi(r_2) \ge \psi(r_1 + r_2)$ for all $r_1, r_2 \in \mathbb{R}^n$ (see also [9]).

2.2 Minimal Lifting Functions

We are now ready to state the main results of the paper. Given a minimal valid function ψ for (3), B_ψ defined in Theorem 3 is a maximal S-free convex set containing f in its interior. Following (4)-(5), it can be uniquely written as $B_\psi = \{x \in \mathbb{R}^n \,|\, a_i(x-f) \le 1,\, i \in I\}$ and so $\psi(r) = \max_{i\in I} a_i r$.

For every $r \in \mathbb{R}^n$, let $I(r) = \{i \in I \,|\, \psi(r) = a_i r\}$. Given $x \in \mathbb{R}^n$, let

$$R(x) := \{r \in \mathbb{R}^n \,|\, I(r) \supseteq I(x-f) \text{ and } I(x-f-r) \supseteq I(x-f)\}.$$

We define

$$R_\psi := \bigcup_{x \in S\cap B_\psi} R(x).$$

Figure 1 illustrates the region R_ψ for several examples. Our first result is the following.

Theorem 4. *Let ψ be a minimal valid function for* (3)*. If π is a minimal lifting of ψ, then $\pi(r) = \psi(r)$ for every $r \in R_\psi$.*
Conversely, for every $r \notin R_\psi$, there exists a lifting π of ψ such that $\pi(r) < \psi(r)$.

The proof of Theorem 4 will be given in Section 4. The second result in this paper considers the case $S = \mathbb{Z}^n$. In this case, we give a necessary and sufficient condition for the existence of a *unique* minimal lifting function.

Theorem 5. *Let ψ be a minimal valid function for* (3) *with $S = \mathbb{Z}^n$. There exists a unique minimal lifting π of ψ if and only if $R_\psi + \mathbb{Z}^n$ covers all of $\mathbb{R}^n$.*

The proof of Theorem 5 will be given in Section 5. We conclude this section by presenting properties of the regions $R(x)$, R_ψ and of minimal liftings. These properties will be used in the remainder of the paper.

Proposition 6. *Let ψ be a minimal valid function for* (3)*. For every $x \in \mathbb{R}^n$, $R(x) = \{r \in \mathbb{R}^n \,|\, \psi(r) + \psi(x-f-r) = \psi(x-f)\}$.*

Proof. We can uniquely write $B_\psi = \{x \in \mathbb{R}^n \,|\, a_i(x-f) \le 1,\, i \in I\}$. Let $h \in I(x-f)$. Then

$$\psi(x-f) = a_h(x-f) = a_h r + a_h(x-f-r) \le \max_{i\in I} a_i r + \max_{i \in I} a_i(x-f-r) = \psi(r) + \psi(x-f-r).$$

In the above expression, equality holds if and only if $h \in I(r)$ and $h \in I(x-f-r)$. □

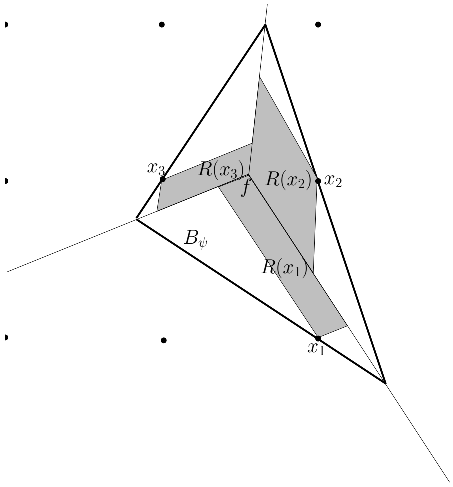
Proposition 7. *Let ψ be a minimal valid function for* (3)*. For every $x \in \mathbb{R}^n$, the region $R(x)$ is a polyhedron, namely*

$$R(x) = \{r \in \mathbb{R}^n \,|\, a_i r + a_j(x-f-r) \le \psi(x-f) \text{ for all } i,j \in I\}.$$

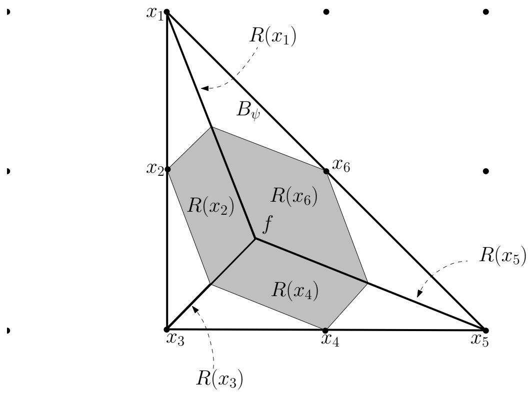
Proof. Follows directly from the proof of Proposition 6. □

Proposition 8. *Let ψ be a minimal valid function for* (3) *and let $B_\psi = \{x \in \mathbb{R}^n : \, a_i(x-f) \le 1,\, i \in I\}$. Let $L_\psi := \{r \in \mathbb{R}^n \,|\, a_i r = a_j r$ for all $i, j \in I\}$. Then*

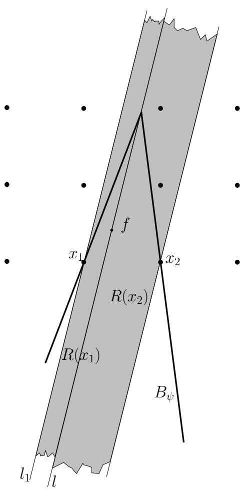
i) $\operatorname{rec}(R(x)) = \operatorname{rec}(R_\psi) = L_\psi$ *for every $x \in \mathbb{R}^n$.*



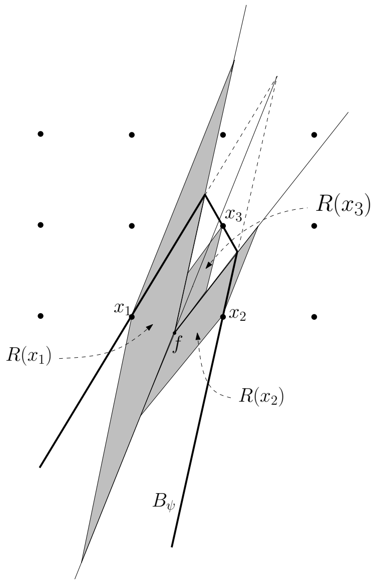
(a) A maximal $\mathbb{Z}2$-free triangle with three integer points



(b) A maximal $\mathbb{Z}2$-free triangle with integer vertices



(c) A wedge



(d) A truncated wedge

Figure 1: Regions $R(x)$ for some maximal S-free convex sets B in the plane and for $x \in S \cap B$. The thick dark line indicates the boundary of B_ψ. For a particular x, the dark gray regions denote $R(x)$ translated by f. The jagged lines in a region indicate that it extends to infinity. For example, in Figure 1(c), $R(x_1)$ is the strip between lines l_1 and l. Figure 1(b) shows an example where $R(x)$ is full-dimensional for x_2, x_4, x_6, but is not full-dimensional for x_1, x_3, x_5.

ii) $R(x) = R(x')$ *for every* $x, x' \in R^n$ *such that* $x - x' \in L_\psi$.

iii) R_ψ *is a finite union of polyhedra.*

iv) L_ψ *is contained in the interior of* R_ψ.

Proof. *i)* By Proposition 7, $R(x) = \{r \in \mathbb{R}^n \,|\, a_i r + a_j(x - f - r) \leq \psi(x - f) \text{ for all } i, j \in I\}$. Thus a vector $r \in \mathbb{R}^n$ belongs to $\text{rec}(R(x))$ if and only if $a_i r - a_j r \leq 0$ for all $i, j \in I$. The latter condition is verified if and only if, for all $i, j \in I$, r satisfies $a_i r - a_j r \leq 0$ and $a_j r - a_i r \leq 0$, that is, if and only if $r \in L_\psi$. This shows that $\text{rec}(R(x)) = L_\psi$ for every $x \in \mathbb{R}^n$, and consequently $\text{rec}(R_\psi) = L_\psi$.

ii) If $x - x' \in L_\psi$, then, for all $i \in I$, $a_i(x - x') = \alpha$ for some constant α. For all $r \in \mathbb{R}^n$, it follows that $a_i(x - f - r) = a_i(x - x') + a_i(x' - f - r) = \alpha + a_i(x' - f - r)$. Hence $\psi(x - f - r) = \alpha + \psi(x' - f - r)$ and, for $i \in I$, $\psi(x - f - r) = a_i(x - f - r)$ if and only if $\psi(x' - f - r) = a_i(x' - f - r)$. Thus $I(x - f - r) = I(x' - f - r)$ for all $r \in \mathbb{R}^n$, and in particular $I(x - f) = I(x' - f)$. It follows that $I(x - f) \subseteq I(r) \cap I(x - f - r)$ if and only if $I(x' - f) \subseteq I(r) \cap I(x' - f - r)$. That is, $r \in R(x)$ if and only if $r \in R(x')$.

iii) Let $B := B_\psi$ and let $L := \text{rec}(B \cap \text{conv}(S))$. By Theorem 1, L is a rational subspace contained in $\text{lin}(B)$. Since $\text{lin}(B) = \{r \in \mathbb{R}^n \,|\, a_i r = 0 \text{ for all } i \in I\}$, we have that $L \subseteq \text{lin}(B) \subseteq L_\psi$.

Let P be the orthogonal projection of B onto the null-space $L^\perp$ of L, and let Q be the projection of $\text{conv}(S)$ onto $L^\perp$. Since $L = \text{rec}(B \cap \text{conv}(S))$, it follows that $P \cap Q$ is a polytope.

Given two elements $x, x' \in S$ whose orthogonal projections onto $L^\perp$ coincide, it follows that $x - x' \in L \subseteq L_\psi$, and therefore by *ii)* $R(x) = R(x')$. It follows that the number of sets $R(x)$, $x \in S \cap B$, is at most the cardinality of the orthogonal projection $\tilde{S}$ of $S \cap B$ onto $L^\perp$.

Let Λ be the orthogonal projection of $\mathbb{Z}^n$ onto $L^\perp$. Since L is a rational space, it follows that Λ is a lattice (see for example [5]). In particular, there exists $\varepsilon > 0$ such that $\|y - z\| \geq \varepsilon$ for all $y, z \in \Lambda$. Since $P \cap Q$ is a polytope, it follows that $P \cap Q \cap \Lambda$ is finite. Since $\tilde{S} \subseteq P \cap Q \cap \Lambda$, it follows that $\tilde{S}$ is a finite set.

We conclude that the family of polyhedra $R(x)$, $x \in S \cap B$, has a finite number of elements, thus $R_\psi = \bigcup_{x \in S \cap B} R(x)$ is the union of a finite number of polyhedra.

iv) By *i)*, for every $r \in R_\psi$, $\{r\} + L_\psi$ is contained in R_ψ. It was proved in [13] that R_ψ contains the origin in its interior. That is, there exists $\varepsilon > 0$ such that $r \in R_\psi$ for all r such that $\|r\| < \varepsilon$. It then follows that $\{r \,|\, \|r\| < \varepsilon\} + L_\psi \subseteq R_\psi$, hence L_ψ is contained in the interior of R_ψ. □

Proposition 9. *Let* ψ *be a minimal valid function for* (3). *Every minimal lifting for* ψ *is a continuous function.*

Proof. Let π be a minimal lifting of ψ. It means that the function (ψ, π) is a minimal valid function for (2). It is known that, if (ψ, π) is a minimal valid function for (2), then π is subadditive. The latter fact was shown by Johnson [18] for the case $S = \mathbb{Z}^n$, but the proof for the general case is identical (see also [12] for a proof). To prove that π is continuous, we need to show that, given $\tilde{r} \in \mathbb{R}^n$, for every $\delta > 0$ there exists $\varepsilon > 0$ such that $|\pi(r) - \pi(\tilde{r})| < \delta$

for all $r \in \mathbb{R}^n$ satisfying $\|r - \tilde{r}\| < \varepsilon$. Since ψ is a continuous function, $\psi(0) = 0$ and ψ coincides with π in some open ball containing the origin, it follows that, for every $\delta > 0$, there exists $\varepsilon > 0$ such that, for all $r \in \mathbb{R}^n$ satisfying $\|r - \tilde{r}\| < \varepsilon$, we have $|\psi(r - \tilde{r})| < \delta$, $\pi(r - \tilde{r}) = \psi(r - \tilde{r})$, and $\pi(\tilde{r} - r) = \psi(\tilde{r} - r)$. Hence, for all $r \in \mathbb{R}^n$ satisfying $\|r - \tilde{r}\| < \varepsilon$,

$$|\pi(r) - \pi(\tilde{r})| \leq \max\{\pi(\tilde{r} - r), \pi(r - \tilde{r})\} = \max\{\psi(r - \tilde{r}), \psi(\tilde{r} - r)\} < \delta,$$

where the first inequality follows from the subadditivity of π, since $\pi(r) \leq \pi(\tilde{r}) + \pi(r - \tilde{r})$ and $\pi(\tilde{r}) \leq \pi(r) + \pi(\tilde{r} - r)$. □

3 Minimum lifting coefficient of a single variable

Given $r^* \in \mathbb{R}^n$, we consider the set of solutions to

$$\begin{aligned} x = \quad & f + \sum_{r \in \mathbb{R}^n} r s_r + r^* y_{r^*} \\ & x \in S \\ & s \geq 0 \\ & y_{r^*} \geq 0,\ y_{r^*} \in \mathbb{Z} \\ & s \text{ has finite support.} \end{aligned} \tag{6}$$

Given a minimal valid function ψ for (3) and scalar λ, we say that the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda y_{r^*} \geq 1$ is valid for (6) if it holds for every (x, s, y_{r^*}) satisfying (6). We denote by $\pi^*(r^*)$ the minimum value of λ for which $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda y_{r^*} \geq 1$ is valid for (6).

We observe that, for *any* lifting π of ψ, we have

$$\pi^*(r^*) \leq \pi(r^*).$$

Indeed, $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \pi(r^*) y_{r^*} \geq 1$ is valid for (6), since, for any $(\bar{s}, \bar{y}_{r^*})$ satisfying (6), the vector $(\bar{s}, \bar{y})$, defined by $\bar{y}_r = 0$ for all $r \in \mathbb{R}^n \setminus \{r^*\}$, satisfies (2).

The following is easy to show (see [13]).

Lemma 10. *If ψ is a minimal valid function for* (3) *and π is a minimal lifting of ψ, then $\pi \leq \psi$.*

So we have the following relation for every minimal lifting π of ψ :

$$\pi^*(r) \leq \pi(r) \leq \psi(r) \quad \text{for all } r \in R^n.$$

In general π^* is not a lifting of ψ, but if it is, then the above relation implies that it is the unique minimal lifting of ψ.

Lemma 11. *For any $r \in \mathbb{R}^n$ such that $\pi^*(r) = \psi(r)$, we have $\pi(r) = \psi(r)$ for every minimal lifting π of ψ. Conversely, if $\pi^*(r^*) < \psi(r^*)$ for some $r^* \in \mathbb{R}^n$, then there exists some minimal lifting π of ψ such that $\pi(r^*) = \pi^*(r^*) < \psi(r^*)$.*

Proof. The first part follows from $\pi^* \leq \pi \leq \psi$. For the second part, given $r^* \in \mathbb{R}^n$ such that $\pi^*(r^*) < \psi(r^*)$, we can define a function $\pi' : \mathbb{R}^n \to \mathbb{R}$ by $\pi'(r^*) = \pi^*(r^*)$ and $\pi'(r) = \psi(r)$ for all $r \in \mathbb{R}^n$, $r \neq r^*$. Since ψ is valid for (3), it follows by the definition of $\pi^*(r^*)$ that π' is a lifting of ψ. As observed in the introduction, there exists a minimal lifting π such that $\pi \leq \pi'$. Since $\pi^* \leq \pi$ and $\pi^*(r^*) = \pi'(r^*)$, it follows that $\pi(r^*) = \pi^*(r^*)$. □

Let $r^* \in \mathbb{R}^n$. Given a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i \in I\}$, for any $\lambda \in \mathbb{R}$, we define the set $B(\lambda, r^*) \subset \mathbb{R}^{n+1}$ as follows

$$B(\lambda, r^*) = \{\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \mid a_i(x - f) + (\lambda - a_i r^*)x_{n+1} \leq 1,\ i \in I\}. \tag{7}$$

The following theorem has been proved in [13].

Theorem 12. *Let $r^* \in \mathbb{R}^n$. Given a maximal S-free convex set B with f in its interior, let $\psi = \psi_B$. Given $\lambda \in \mathbb{R}$, the inequality $\sum_{r \in \mathbb{R}^n} \psi(r)s_r + \lambda y_{r^*} \geq 1$ is valid for (6) if and only if $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free.*

Lemma 13. *Let $r^* \in \mathbb{R}^n$, and let B be a maximal S-free convex set with f in its interior. For every λ such that $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free, $B(\lambda, r^*)$ is maximal $S \times \mathbb{Z}_+$-free.*

Proof. Since B is a maximal S-free convex set, then by Theorem 1 each facet of B contains a point $\bar{x}$ of S in its relative interior. Therefore the corresponding facet of $B(\lambda, r^*)$ contains the point $\binom{\bar{x}}{0}$ in its relative interior. If $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free, by Theorem 1 it is a maximal $S \times \mathbb{Z}_+$-free convex set. □

Theorem 14. *Consider $r^* \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We have $\lambda = \pi^*(r^*)$ if and only if $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free and contains a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ such that $\bar{x}_{n+1} > 0$.*

Proof. We first prove that, if $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free and contains a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ such that $\bar{x}_{n+1} > 0$, then $\lambda = \pi^*(r^*)$.

Since $B(\lambda, r^*)$ is $S \times \mathbb{Z}_+$-free, Theorem 12 and the definition of π^* imply that $\lambda \geq \pi^*(r^*)$. We claim that $\binom{\bar{x}}{\bar{x}_{n+1}}$ is in the interior of $B(\lambda - \epsilon, r^*)$ for every $\epsilon > 0$. Indeed, for all $i \in I$, $a_i(\bar{x} - f) + (\lambda - \epsilon - a_i r^*)\bar{x}_{n+1} = a_i(\bar{x} - f) + (\lambda - a_i r^*)\bar{x}_{n+1} - \epsilon\bar{x}_{n+1} < 1$, where the inequality holds because $\binom{\bar{x}}{\bar{x}_{n+1}} \in B(\lambda, r^*)$ implies that $a_i(x - f) + (\lambda - a_i r^*)x_{n+1} \leq 1$ and $\bar{x}_{n+1} > 0$ implies $\epsilon\bar{x}_{n+1} > 0$. This proves our claim. Since $\binom{\bar{x}}{\bar{x}_{n+1}}$ is in the interior of $B(\lambda - \epsilon, r^*)$ for every $\epsilon > 0$, Theorem 12 and the definition of π^* imply that $\lambda - \epsilon < \pi^*(r^*)$. Hence $\lambda = \pi^*(r^*)$.

We now prove the converse. Let $\lambda^* = \pi^*(r^*)$. The definition of π^* and Theorem 12 imply that $B(\lambda^*, r^*)$ is $S \times \mathbb{Z}_+$-free. It only remains to show that $B(\lambda^*, r^*)$ contains a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ such that $\bar{x}_{n+1} > 0$.

Note that, for all $\lambda \in \mathbb{R}$, $B(\lambda, r^*) \cap (\mathbb{R}^n \times \{0\}) = B \times \{0\}$.

We consider two possible cases: either there exists $\binom{\bar{r}}{\bar{r}_{n+1}} \in \text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+))$ such that $\bar{r}_{n+1} > 0$, or $\text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(B \cap \text{conv}(S)) \times \{0\}$.

Case 1. There exists $\binom{\bar{r}}{\bar{r}_{n+1}} \in \text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+))$ such that $\bar{r}_{n+1} > 0$.

Note that $\text{rec}(\text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(\text{conv}(S)) \times \mathbb{R}_+$, thus $\bar{r} \in \text{rec}(\text{conv}(S))$. By Lemma 13, $B(\lambda^*, r^*)$ is a maximal $S \times \mathbb{Z}_+$-free convex set, thus by Theorem 1, $\text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+))$ is rational. Hence, we can choose $\binom{\bar{r}}{\bar{r}_{n+1}}$ integral. Since B is a maximal S-free convex set, there exists $\tilde{x} \in S \cap B$. Let $\binom{\bar{x}}{\bar{x}_{n+1}} = \binom{\tilde{x}}{0} + \binom{\bar{r}}{\bar{r}_{n+1}}$. Since $\bar{r} \in \text{rec}(\text{conv}(S))$ and $\tilde{x} \in S$, it follows that $\bar{x} \in \text{conv}(S)$. Since $\bar{x} \in \mathbb{Z}^n$, we conclude that $\bar{x} \in S$. Furthermore, $\bar{x}_{n+1} = \bar{r}_{n+1}$, thus $\bar{x}_{n+1} \in \mathbb{Z}$ and $\bar{x}_{n+1} > 0$. Finally, since $\binom{\tilde{x}}{0} \in B(\lambda^*, r^*)$ and $\binom{\bar{r}}{\bar{r}_{n+1}} \in \text{rec}(B(\lambda^*, r^*))$, we conclude that $\binom{\bar{x}}{\bar{x}_{n+1}} \in B(\pi^*(r^*), r^*)$.

Case 2. $\text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(B \cap \text{conv}(S)) \times \{0\}$.

Claim. $\exists\, \bar{\varepsilon} > 0$ such that $\text{rec}(B(\lambda^* - \bar{\varepsilon}, r^*) \cap \text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(B \cap \text{conv}(S)) \times \{0\}$.

Since $\text{conv}(S)$ is a polyhedron, $\text{conv}(S) = \{x \in \mathbb{R}^n \mid Cx \leq d\}$ for some matrix (C, d). By assumption, there is no vector $\binom{r}{1}$ in $\text{rec}(B(\lambda^*, r^*) \cap \text{conv}(S \times \mathbb{Z}_+))$. Thus the system

$$\begin{aligned} a_i r + (\lambda^* - a_i r^*) &\leq 0, \quad i \in I \\ Cr &\leq 0 \end{aligned}$$

is infeasible. By Farkas' Lemma, there exist scalars $\mu_i \geq 0$, $i \in I$ and a nonnegative vector γ such that

$$\begin{aligned} \sum_{i \in I} \mu_i a_i + \gamma C &= 0 \\ \lambda^* (\sum_{i \in I} \mu_i) - (\sum_{i \in I} \mu_i a_i) r^* &> 0. \end{aligned}$$

This implies that there exists some $\bar{\varepsilon} > 0$ small enough such that

$$\begin{aligned} \sum_{i \in I} \mu_i a_i + \gamma C &= 0 \\ (\lambda^* - \bar{\varepsilon})(\sum_{i \in I} \mu_i) - (\sum_{i \in I} \mu_i a_i) r^* &> 0, \end{aligned}$$

thus the system

$$\begin{aligned} a_i r + (\lambda^* - \bar{\varepsilon} - a_i r^*) &\leq 0, \quad i \in I \\ Cr &\leq 0 \end{aligned}$$

is infeasible. This implies that $\text{rec}(B(\lambda^* - \bar{\varepsilon}, r^*) \cap \text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(B \cap \text{conv}(S)) \times \{0\}$.

By the claim, there exists $\bar{\varepsilon}$ such that $\text{rec}(B(\lambda^* - \bar{\varepsilon}, r^*) \cap \text{conv}(S \times \mathbb{Z}_+)) = \text{rec}(B \cap \text{conv}(S)) \times \{0\}$. This implies that there exists a scalar M such that $\bar{x}_{n+1} \leq M$ for every point $\binom{\bar{x}}{\bar{x}_{n+1}} \in B(\lambda^* - \bar{\varepsilon}, r^*) \cap (S \times \mathbb{Z}_+)$.

Remark 2 and Lemma 13 imply that there exists $\delta > 0$ such that, for every $\binom{\bar{x}}{\bar{x}_{n+1}} \in (S \times \mathbb{Z}_+) \setminus B(\lambda^*, r^*)$, there exists $h \in I$ such that $a_h(\bar{x} - f) + (\lambda^* - a_h r^*)\bar{x}_{n+1} \geq 1 + \delta$. Choose $\varepsilon > 0$ such that $\varepsilon \leq \bar{\varepsilon}$ and $\varepsilon M \leq \delta$.

Since $\pi^*(r^*) = \lambda^*$, Theorem 12 implies that $B(\lambda^* - \varepsilon, r^*)$ has a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$ in its interior. Thus $a_i(\bar{x} - f) + (\lambda^* - \varepsilon - a_i r^*)\bar{x}_{n+1} < 1$, $i \in I$.

We show that $\binom{\bar{x}}{\bar{x}_{n+1}}$ is also in $B(\lambda^*, r^*)$. Suppose not. Then, by our choice of δ, there exists $h \in I$ such that $a_h(\bar{x} - f) + (\lambda^* - a_h r^*)\bar{x}_{n+1} \geq 1 + \delta$.

It follows from the definition of $B(\lambda, r^*)$ given in (7) that $B(\lambda^* - \varepsilon, r^*) \cap (\mathbb{R}^n \times \mathbb{R}_+) \subseteq B(\lambda^* - \bar{\varepsilon}, r^*) \cap (\mathbb{R}^n \times \mathbb{R}_+)$ because $\varepsilon \leq \bar{\varepsilon}$; therefore $\bar{x}_{n+1} \leq M$. Hence

$$1+\delta \leq a_h(\bar{x}-f)+(\lambda^*-a_h r^*)\bar{x}_{n+1} \leq a_h(\bar{x}-f)+(\lambda^*-\varepsilon-a_h r^*)\bar{x}_{n+1}+\varepsilon M < 1+\varepsilon M \leq 1+\delta,$$

a contradiction.

Hence $\binom{\bar{x}}{\bar{x}_{n+1}}$ is in $B(\lambda^*, r^*)$. Since B is S-free and $B(\lambda^* - \varepsilon, r^*) \cap (\mathbb{R}^n \times \{0\}) = B \times \{0\}$, it follows that $B(\lambda^* - \varepsilon, r^*)$ does not contain any point of $S \times \{0\}$ in its interior. Thus $\bar{x}_{n+1} > 0$. □

4 Proof of Theorem 4

By Lemma 11, the statement of Theorem 4 is equivalent to the equality $R_\psi = \{r \in \mathbb{R}^n : \pi^*(r) = \psi(r)\}$. Moreover, by Proposition 6, it can be restated as the equivalence between i) and ii):

i) $\pi^*(r^*) = \psi(r^*)$.

ii) There exists $x \in S \cap B_\psi$ such that $\psi(r^*) + \psi(x - f - r^*) = \psi(x - f) = 1$.

We first prove ii) implies i). The vector (s, y_{r^*}) defined by $y_{r^*} = 1$, $s_{x-f-r^*} = 1$, $s_r = 0$ for all $r \neq x - f - r^*$, is a solution of (6). Since $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \pi^*(r^*)y_{r^*} \geq 1$ is valid for (6), it follows that

$$1 \leq \pi^*(r^*) + \psi(x - f - r^*) \leq \psi(r^*) + \psi(x - f - r^*) = 1,$$

thus $\pi^*(r^*) = \psi(r^*)$.

We now prove i) implies ii). By Theorem 14, $B(\pi^*(r^*), r^*)$ contains a point $\binom{x}{x_{n+1}} \in S \times \mathbb{Z}_+$ such that $x_{n+1} > 0$. Since $\pi^*(r^*) = \psi(r^*) = \max_{i\in I} a_i r^*$, the coefficients of x_{n+1} in the inequalities (7) defining $B(\pi(r^*), r^*)$ are all nonnegative. This shows that $B(\pi^*(r^*), r^*)$ contains the point $\binom{x}{1}$ and therefore $a_i(x-f)+\psi(r^*)-a_i r^* \leq 1$ for all $i \in I$. This implies that $\max_{i\in I} a_i(x-f-r^*)+\psi(r^*) \leq 1$. Hence we have that $\psi(x-f-r^*)+\psi(r^*) \leq 1$. Conversely, we also have $1 \leq \psi(x - f) \leq \psi(x - f - r^*) + \psi(r^*)$, where the first inequality follows from the fact that $x \in S$ and that, by Theorem 3, B_ψ is S-free, while the second inequality follows from the fact that ψ is subadditive. We conclude that $\psi(r^*) + \psi(x - f - r^*) = \psi(x - f) = 1$.

5 Proof of Theorem 5

Let ψ be a minimal valid function for (3) and let $B := B_\psi = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\, i \in I\}$. In this section we assume $S = \mathbb{Z}^n$. Under this assumption, Theorem 1 shows that:

- B is a maximal lattice-free convex set.
- $\text{rec}(B) = \text{lin}(B) = \{r \in \mathbb{R}^n \mid a_i r = 0 \text{ for all } i \in I\}$ and $\text{rec}(B)$ is a rational subspace.
- $\psi(r) = \max_{i\in I} a_i r \geq 0$ for every $r \in \mathbb{R}^n$.

Let $L_\psi := \{r \in \mathbb{R}^n \,|\, a_i r = a_j r \text{ for all } i, j \in I\}$. To prove Theorem 5, we need the following three lemmas.

Lemma 15. *$L_\psi = \text{lin}(B)$. Furthermore, $R_\psi + \mathbb{Z}^n$ is a closed set.*

Proof. Let $L := \text{lin}(B)$. It follows from Theorem 1 that L is a rational linear subspace and $\text{rec}(B) = L$. Given $r \in L_\psi$, it follows that $a_i r = a_j r$ for all $i, j \in I$, which implies that $a_i r = \alpha$ for all $i \in I$ for some constant α that only depends on r. If $\alpha \leq 0$ then $r \in \text{rec}(B) = L$, while if $\alpha \geq 0$ then $-r \in \text{rec}(B) = L$, thus $r \in L$. The converse inclusion $L \subseteq L_\psi$ is trivial.

Next we show that $R_\psi + \mathbb{Z}^n$ is a closed set. It follows from Proposition 8 that R_ψ is the union of a finite number of polyhedra, $R_1, \ldots, R_k$, and that $\text{rec}(R_i) = L$ for $i = 1, \ldots, k$. Let $\tilde{R}_i$ be the orthogonal projection of R_i onto $L^\perp$. It follows that $\tilde{R}_i$ is a polytope, so in particular $\tilde{R}_i$ is compact. Furthermore, $R_i = \tilde{R}_i + L$. Let Λ be the orthogonal projection of $\mathbb{Z}^n$ onto $L^\perp$. Since L is rational, Λ is a lattice, so in particular it is closed. Note that $R_i + \mathbb{Z}^n = (\tilde{R}_i + \Lambda) + L$. Since the Minkowski sum of a compact set and a closed set is closed (see e.g. [1] Lemma 5.3 (4)), it follows that $\tilde{R}_i + \Lambda$ is closed. Since the Minkowski sum of a closed set and a linear subspace is closed, it follows that $(\tilde{R}_i + \Lambda) + L$ is closed. □

Lemma 16. *Let $H := \mathbb{R}^n \setminus (R_\psi + \mathbb{Z}^n)$. If $H \neq \emptyset$, then there exists $\bar{r} \in R_\psi$ such that $\bar{r}$ belongs to the closure of H. Any such vector satisfies $\psi(\bar{r}) > 0$.*

Proof. By Lemma 15, $R_\psi + \mathbb{Z}^n$ is closed. It follows that, if $R_\psi + \mathbb{Z}^n \neq \mathbb{R}^n$, then there exists a point $\tilde{r} \in R_\psi + \mathbb{Z}^n$ such that $\tilde{r}$ belongs to the closure of H. Since $\tilde{r} \in R_\psi + \mathbb{Z}^n$, there exists $\bar{r} \in R_\psi$ and $w \in \mathbb{Z}^n$ such that $\tilde{r} = \bar{r} + w$. We show that the point $\bar{r}$ belongs to the closure of H. If not, then there exists an open ball C centered at $\bar{r}$ contained in the interior of R_ψ, thus $C + w$ is an open ball centered at $\tilde{r}$ contained in the interior of $R_\psi + \mathbb{Z}^n$, contradicting the fact that $\tilde{r}$ is in the closure of H.

Finally, given $\bar{r}$ on the boundary of R_ψ, Proposition 8 iv) and the fact that $L_\psi = \text{lin}(B)$ imply that $\bar{r} \notin \text{lin}(B)$, thus $\psi(\bar{r}) > 0$. □

Lemma 17. *Let $r^* \in \mathbb{R}^n$. If $\pi^*(r^*) > 0$, then $\text{rec}(B(\pi^*(r^*), r^*) \cap (\mathbb{R}^n \times \mathbb{R}_+)) = \text{rec}(B) \times \{0\}$.*

Proof. Let $\lambda^* = \pi^*(r^*) > 0$. Let $\binom{r}{r_{n+1}} \in \mathbb{R}^n \times \mathbb{R}_+$ such that $\binom{r}{r_{n+1}} \in \text{rec}(B(\lambda^*, r^*))$. If $r_{n+1} = 0$, it follows that $r \in \text{rec}(B)$, thus $\binom{r}{r_{n+1}} \in \text{rec}(B) \times \{0\}$. So suppose by contradiction that $r_{n+1} > 0$. We may assume without loss of generality that $r_{n+1} = 1$. Since $B(\lambda^*, r^*)$ is a maximal $\mathbb{Z}^n \times \mathbb{Z}_+$-free convex set, it follows from Theorem 1 that $a_i r + (\lambda^* - a_i r^*) = 0$ for all $i \in I$. Since $\lambda^* > 0$, the point $\hat{r} := \frac{r^* - r}{\lambda^*}$ satisfies $a_i \hat{r} = 1$ for all $i \in I$. It follows that $\hat{r} \notin \text{rec}(B)$ while $-\hat{r} \in \text{rec}(B)$, contradicting the fact that $\text{rec}(B) = \text{lin}(B)$. □

Proof of Theorem 5. As already mentioned in the introduction, it is shown in [13] that if $R_\psi + \mathbb{Z}^n = \mathbb{R}^n$ then π^* is the unique minimal valid lifting of ψ.

We show the converse, that is, we show that if $R_\psi + \mathbb{Z}^n$ *does not* cover all of $\mathbb{R}^n$, then there exist at least two distinct minimal liftings of ψ.

We first observe that there exist two distinct minimal liftings of ψ if and only if π^* is not a lifting for ψ. Indeed, if π^* is a lifting for ψ, then π^* is the unique minimal lifting for ψ, since any other lifting π satisfies $\pi^* \leq \pi$. Conversely, suppose that π^* is not a lifting for ψ. Given any minimal lifting π of ψ, since π^* is not valid it follows that there exists $r' \in \mathbb{R}^n$ such that $\pi^*(r') < \pi(r')$. By Lemma 11, there exists a minimal lifting π' such that $\pi'(r') = \pi^*(r')$. Thus π and π' are distinct minimal liftings of ψ.

Thus, we only need to show that, if $R_\psi + \mathbb{Z}^n \neq \mathbb{R}^n$, then π^* is not valid. Suppose by contradiction that $R_\psi + \mathbb{Z}^n \neq \mathbb{R}^n$ but π^* is a lifting of ψ. It follows that π^* is a minimal lifting for ψ, thus by Proposition 9, π^* is a continuous function.

Let $H = \mathbb{R}^n \setminus (R_\psi + \mathbb{Z}^n)$. We will show the following.

Claim 1. *There exists $p \in H$ such that $B(\pi^*(p), p)$ contains a point $\binom{\bar{x}}{1}$ with $\bar{x} \in \mathbb{Z}^n$.*

Before proving Claim 1, we use it to conclude the proof of Theorem 5. By Claim 1, there exists $p \in \mathbb{R}^n \setminus (R_\psi + \mathbb{Z}^n)$ such that $B(\pi^*(p), p)$ contains a point $\binom{\bar{x}}{1}$ with $\bar{x} \in \mathbb{Z}^n$. It follows that $a_i(\bar{x} - f) + (\pi^*(p) - a_i p) \leq 1$ for all $i \in I$. Moreover, by Theorem 12, $B(\pi^*(p), p)$ is $S \times \mathbb{Z}_+$-free, thus there exists $h \in I$ such that $a_h(\bar{x} - f) + (\pi^*(p) - a_h p) = 1$. This shows that $a_i(\bar{x} - f - p) \leq 1 - \pi^*(p)$ for all $i \in I$ and $a_h(\bar{x} - f - p) = 1 - \pi^*(p)$. Since $\psi(\bar{x} - f - p) = \max_{i \in I} a_i(\bar{x} - f - p) = a_h(\bar{x} - f - p)$, we have that

$$\psi(\bar{x} - f - p) = 1 - \pi^*(p).$$

Since π^* is a lifting, $\pi^*(p) + \pi^*(\bar{x} - f - p) \geq 1$ and since $\pi^*(\bar{x} - f - p) \leq \psi(\bar{x} - f - p)$, this shows that $\pi^*(\bar{x} - f - p) = \psi(\bar{x} - f - p)$. It follows that $\bar{x} - f - p \in R_\psi$. In particular, $\bar{x} - f - p \in R(\tilde{x})$ for some $\tilde{x} \in \mathbb{Z}^n \cap B$. By definition of $R(\tilde{x})$, $\tilde{x} - f - (\bar{x} - f - p) \in R(\tilde{x})$, that is, $\tilde{x} - \bar{x} + p \in R(\tilde{x})$. This implies that $p \in R(\tilde{x}) + \mathbb{Z}^n$, a contradiction.

The remainder of the proof is devoted to showing Claim 1. By Lemma 16, the closure of H contains a point $\bar{r} \in R_\psi$ and such a vector satisfies $\pi^*(\bar{r}) = \psi(\bar{r}) > 0$. Since π^* is continuous, there exists $\bar{\varepsilon} > 0$ such that, for every $r \in \mathbb{R}^n$ satisfying $\|r - \bar{r}\| \leq \bar{\varepsilon}$, $\pi^*(r) > 0$.

Let $\mu = \min\{\pi^*(r) \mid \|r - \bar{r}\| \leq \bar{\varepsilon}\}$. Note that μ is well defined since π^* is continuous and $\{r \in \mathbb{R}^n \mid \|r - \bar{r}\| \leq \bar{\varepsilon}\}$ is compact. Furthermore, by the choice of $\bar{\varepsilon}$, $\mu > 0$. Let $M = \mu^{-1}$.

Claim 2. *For every $r \in \mathbb{R}^n$ satisfying $\|r - \bar{r}\| < \bar{\varepsilon}$ and every $\binom{x}{x_{n+1}} \in B(\pi^*(r), r)$, we have $x_{n+1} \leq M$.*

Let $r \in \mathbb{R}^n$ such that $\|r - \bar{r}\| < \bar{\varepsilon}$. By Lemma 17, $\max\{x_{n+1} \mid \binom{x}{x_{n+1}} \in B(\pi^*(r), r)\}$ is bounded. The above is a linear program, thus the set of its optimal solutions contains a minimal face of $B(\pi^*(r), r)$. Since the point $\binom{f}{0} + \frac{1}{\pi^*(r)}\binom{r}{1}$ satisfies all inequalities $a_i(x - f) + (\pi^*(r) - a_i r)x_{n+1} \leq 1$ at equality, it follows that $B(\pi^*(r), r)$ has a unique minimal face and that $\binom{f}{0} + \frac{1}{\pi^*(r)}\binom{r}{1}$ is in it. It follows that $\binom{f}{0} + \frac{1}{\pi^*(r)}\binom{r}{1}$ is an optimal solution, thus $\max\{x_{n+1} \mid \binom{x}{x_{n+1}} \in B(\pi^*(r), r))\} = \frac{1}{\pi^*(r)} \leq M$.

By Remark 2 and Lemma 13, there exists $\delta > 0$ such that, for every $\binom{\bar{x}}{\bar{x}_{n+1}} \in (\mathbb{Z}^n \times \mathbb{Z}_+) \setminus B(\psi(\bar{r}), \bar{r})$, $a_i(\bar{x} - f) + (\psi(\bar{r}) - a_i \bar{r})\bar{x}_{n+1} \geq 1 + \delta$, for some $i \in I$.

By Lemma 15, H is an open set. Thus, since π^* is a continuous function, $\pi^*(\bar{r}) = \psi(\bar{r})$ and $\bar{r}$ is on the closure of H, there exists $p \in H$ such that $\|p - \bar{r}\| < \bar{\varepsilon}$ and

$$|(\psi(\bar{r}) - a_i \bar{r}) - (\pi^*(p) - a_i p)| < \frac{\delta}{M} \text{ for all } i \in I.$$

By Theorem 14 and Claim 2, $B(\pi^*(p), p)$ contains a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in \mathbb{Z}^n \times \mathbb{Z}_+$ such that $0 < \bar{x}_{n+1} \leq M$. We conclude by showing that $\binom{\bar{x}}{1}$ is in $B(\pi^*(p), p)$, thus proving Claim 1.

First we show that $\binom{\bar{x}}{1}$ is in $B(\psi(\bar{r}), \bar{r})$. Indeed, for all $i \in I$,

$$\begin{array}{rcl} a_i(\bar{x} - f) + (\psi(\bar{r}) - a_i\bar{r}) & \leq & a_i(\bar{x} - f) + (\psi(\bar{r}) - a_i\bar{r})\bar{x}_{n+1} \\ & < & a_i(\bar{x} - f) + (\pi^*(p) - a_i p + \frac{\delta}{M})\bar{x}_{n+1} \\ & \leq & 1 + \delta, \end{array} \tag{8}$$

where the first inequality follows from the facts that $\psi(\bar{r}) \geq a_i\bar{r}$ and $\bar{x}_{n+1} \geq 1$, while the last inequality follows from the facts that $\binom{\bar{x}}{\bar{x}_{n+1}} \in B(\pi^*(p), p)$ and $\bar{x}_{n+1} \leq M$. By our choice of δ, the strict inequality in (8) shows that $\binom{\bar{x}}{1}$ is in $B(\psi(\bar{r}), \bar{r})$. In particular, since $\psi(\bar{r}) \geq a_i\bar{r}$ for all $i \in I$, it follows that $a_i(\bar{x} - f) \leq 1$ for all $i \in I$.

We finally show that $a_i(\bar{x} - f) + \pi^*(p) - a_i p \leq 1$ for all $i \in I$, implying that $\binom{\bar{x}}{1}$ is in $B(\pi^*(p), p)$. Indeed, if $\pi^*(p) - a_i p \leq 0$, then $a_i(\bar{x} - f) + \pi^*(p) - a_i p \leq 1 + \pi^*(p) - a_i p \leq 1$, while if $\pi^*(p) - a_i p > 0$, then $a_i(\bar{x} - f) + \pi^*(p) - a_i p \leq a_i(\bar{x} - f) + (\pi^*(p) - a_i p)\bar{x}_{n+1} \leq 1$. □

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