HW 4 : Derivative rules.

Power Rule.

We will use the differentiation rules frequently, so it is important to know what they are. Here we will derive them using only algebra and limits. Remember the definition of a derivative: If f is a function, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided this limit exists. The derivative $f'(x)$ is the slope of the tangent line to the graph of f at the point $(x, f(x))$. Let's see an example of how to compute this.

Power rule (first case). What is $f'(x)$ when $f(x) = x^n$, where n is a positive integer? We need to use the formula for expanding $(x + y)^n$:

$$(x+y)^n = x^n + nx^{n-1}y + C_{n-2}x^{n-2}y^2 + \ldots + C_2x^2y^{n-2} + nxy^{n-1} + y^n$$

where the C_i's in the formula are just some constants; we don't need to know what they actually are. Then we calculate the derivative:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x + C_{n-2}x^{n-2}(\Delta x)^2 + \ldots + nx(\Delta x)^{n-1} + (\Delta x)) - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(nx^{n-1} + C_{n-2}x^{n-2}(\Delta x) + \ldots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}}{\Delta x} \\
&= \lim_{\Delta x \to 0} (nx^{n-1} + C_{n-2}x^{n-2}(\Delta x) + \ldots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}) \\
&= nx^{n-1}
\end{aligned}$$

Therefore we've proven the *power rule* for derivatives. If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

We will also show that it works when n is a negative integer. Remember that $x^{-n} = \frac{1}{x^n}$.

Power rule ($n = -1$). What is $f'(x)$ when $f(x) = x^{-1}$? We need to do some algebra manipulation, but it's pretty simple. Let's calculate:

$$\begin{aligned}
\frac{d}{dx}x^{-1} &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x(x+\Delta x))} \\
&= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} \\
&= -\frac{1}{x^2} = -x^{-2}
\end{aligned}$$

Therefore, the power rule *still is true* when we let $n = -1$. If you were to do the same calculation for $f(x) = x^{-2}$, you would get $f'(x) = -2x^{-3}$. You can prove it for any arbitrary negative integer n, and you would get $f'(x) = nx^{n-1}$. You would do exactly the same thing as in the example, except you need to know how to expand $(x + \Delta x)^n$. It turns out that it works also when n is any real number, but this is more complicated to prove.

Recap. The *power rule*: if n is any real number except 0, then

$$\frac{d}{dx}x^n = nx^{n-1}$$

Sum, constant multiple rule.

Sum Rule. Suppose f and g are two differentiable functions. That means $f'(x)$ and $g'(x)$ both exist. What is the derivative of the function h, where $h(x) = f(x) + g(x)$?

$$\begin{aligned}
\frac{d}{dx}(h(x)) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

In words, *the derivative of the sum of functions is the sum of the derivatives.*

1. Exercises.

You do not need to turn these in but are good exercises.

Problem 1. Show that if c is any real number, then $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$ using the definition of a derivative. I'll get you started:

$$\frac{d}{dx}[c \cdot f(x)] = \lim_{\Delta x \to 0} \frac{c \cdot f(x+\Delta x) - c \cdot f(x)}{\Delta x} = ...$$

Problem 2. Same as above, show that $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g(x)$.