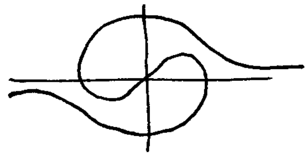
1. (12) For each of the following, give a definition, short (one sentence) answer, or formula.
(a) How can you tell if a curve in $\mathbb{R}^2$ is a graph of a function? Draw a curve which is not the graph of a function, explain why.

~~Answers may vary~~ a graph is a function if it passes the vertical line test; any vertical line intersects the line in <u>at</u> <u>most</u> one point (possibly none).



← not a graph of a function.

(b) Suppose $f(x)$ is defined at all numbers except $x = 0$. Is it possible that $\lim_{x\to 0} f(x)$ exists? If so, give an example. If not, explain why.

Yes: Recall $\lim\limits_{x\to 0} \frac{x^2}{x} = \lim\limits_{x\to 0} x = 0$, but $\frac{x^2}{x}$ is not defined at $x=0$.

Better example: $\lim\limits_{x\to 0} \frac{x^3+x}{x} = \lim\limits_{x\to 0} \frac{x(x^2+1)}{x} = \lim\limits_{x\to 0} x^2+1 = 1$.

Hence: $\lim\limits_{x\to 0} \frac{x^3+x}{x} = 1$ exists, but $\frac{x^3+x}{x}$ is not defined at $x=0$.

(c) What does it mean for a function f to be continuous at a number c? Give the precise definition as seen in class.

f is continuous at c if:

- f is defined at c
- $\lim\limits_{x\to c} f(x)$ exists
- $\lim\limits_{x\to c} f(x) = f(c)$.

(d) Give an example of a function f which is *not* continuous at some number. Explain why it is not continuous.

Example: The function $f(x) = \begin{cases} -1 & x<0 \\ 1 & x\geq 0 \end{cases}$ is not continuous at 0.

Because $\lim\limits_{x\to 0} f(x)$ does not exist.

2. (10 points) Find the intervals of continuity. Wherever the function is *not* continuous, explain why.

(a) $f(x) = \frac{x}{x^2-1}$.

$f(x) = \frac{x}{(x-1)(x+1)}$ - Vertical asymptotes at $x=1$, $x=-1$.

Intervals of continuity: $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

(b) $g(x) = \sqrt{x}/x = x^{-\frac{1}{2}}$

Note: $g(x)$ is not defined for $x \leq 0$. So is continuous on $(0,\infty)$.

(c) $h(t) = \begin{cases} -2 & t \leq -1 \\ 2t & -1 < t \leq 1 \\ \frac{1}{t} & 1 < t \end{cases}$ Need to check $t=-1$ and $t=1$.

$\lim_{t\to -1} h(t) = -2$ and

$f(-1) = -2$

Therefore h is continuous at -1.

$\lim_{t\to 1} h(t)$ DOES NOT EXIST! because

$\lim_{t\to 1^-} h(t) = 2$, $\lim_{t\to 1^+} h(t) = 1$.

The left-hand and right-hand limits do not agree, so $\lim_{t\to 1} h(t)$ does not exist.

$\boxed{\text{So: } h \text{ is continuous on } (-\infty,1) \cup (1,\infty).}$

3.(10) For each function, find (i) the vertical asymptotes and (ii) horizontal asymptotes, if any.
(a) $q(w) = \frac{1}{w^3}$

$\boxed{\text{V.A: at } w=0.}$

H.A: Since degree(1)=0, degree(w^3)=3, we have

$\lim_{w \to \infty} \frac{1}{w^3} = 0.$

$\boxed{\text{So H.A. at } y=0}$

(b) $P(x) = \frac{x+1}{x^2-6x+9}$

Factor denominator: $P(x) = \frac{x+1}{(x-3)^2}$.

V.A: Denom. is 0 at $x=3$, and numerator $\neq 0$ at $x=3$. Therefore $\boxed{\text{V.A. at } x=3}$

H.A: Degree$(x+1) = 1$, Degree$(x^2-6x+9) = 2$.

Since Degree$(x+1) <$ Degree(x^2-6x+9), we have

$\boxed{\text{H.A. at } y=0}$

4. (20 points) Find the derivatives. Use any derivative rules you know, but indicate whenever you use them.

(a) $\frac{d}{dx}[11x^3 + 2.33]$

$$\frac{d}{dx}[11x^3+2.33] = \frac{d}{dx}[11x^3] + \frac{d}{dx}[2.33]$$
$$= 11\cdot(3x^2) + 0$$
$$\boxed{= 33x^2}$$

(b) $\frac{d}{dt}[-16t^2 + t + 100]$

$$\frac{d}{dt}[-16t^2+t+100] = \frac{d}{dt}[-16t^2] + \frac{d}{dt}[t] + \frac{d}{dt}[100]$$
$$= -16\cdot(2t) + 1 + 0$$
$$\boxed{= -32t + 1}$$

(c) $\frac{d}{dx}[\frac{5x^3+2x}{x}]$

Simplify: $\frac{5x^3+2x}{x} = 5x^2+2$

$$\frac{d}{dx}[5x^2+2] = \frac{d}{dx}[5x^2] + \frac{d}{dx}[2]$$
$$\boxed{= 10x}$$

(d) $\sqrt{x} - \frac{x^2+1}{\sqrt{x}}$

Simplify: $\frac{d}{dx}\left[x^{\frac{1}{2}} - x^{2-\frac{1}{2}} - x^{-\frac{1}{2}}\right]$

$= \frac{d}{dx}\left[x^{\frac{1}{2}} - x^{3/2} - x^{-1/2}\right]$

$$\boxed{= \left[\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\right]}$$

(e) $\frac{d}{dx}\left[\frac{(2x+1)^{5/2}}{x(x+1)(2x+1)^{3/2}}\right]$ (Hint: simplify.)

Simplify: $\frac{1}{x(x+1)} \cdot \frac{(2x+1)^{5/2}}{(2x+1)^{3/2}} = \frac{1}{x(x+1)} \cdot (2x+1) = \frac{2x}{x(x+1)} + \frac{1}{x(x+1)}$

$= \frac{2}{x+1} + \frac{1}{x(x+1)}$

$$\frac{d}{dx}\left[\frac{2}{x+1} + \frac{1}{x(x+1)}\right] = \frac{d}{dx}\left[\frac{2}{x+1}\right] + \frac{d}{dx}\left[\frac{1}{x(x+1)}\right]$$

$$= \frac{\frac{d}{dx}(2)\cdot(x+1) - 2\cdot\frac{d}{dx}(x+1)}{(x+1)^2} + \frac{x(x+1)\frac{d}{dx}[1] - 1\cdot\frac{d}{dx}[x(x+1)]}{[x(x+1)]^2}$$

quotient rule

$$= \frac{0\cdot(x+1) - 2\cdot 1}{((x+1)^2} + \frac{x(x+1)\cdot 0 - (2x+1)}{[x(x+1)]^2}$$

$$\boxed{= -\frac{2}{(x+1)^2} - \frac{(2x+1)}{[x(x+1)]^2}}$$

5. (8 points) Suppose I own a home business that produces, oh I don't know, let's say fancy candles. The profit generated from producing x candles is $P(x) = 5x - 100x^{-1}$.

(a) Find the function $P'(x)$.

$$P'(x) = \frac{d}{dx}\left[5x - 100x^{-1}\right] = \boxed{5 + \frac{100}{x^2}}$$

(b). What is the marginal profit for producing 10 fancy candles?

Marginal profit @ 10 candles

$$P'(10) = 5 + \frac{100}{10^2} = 5 + \frac{100}{100} = 5 + 1 = \boxed{6}$$

(c) *Use the marginal profit* to estimate the profit for producing 11 fancy candles. (I *do not* want you to calculate $P(11)$).

$$P(11) \approx P(10) + P'(10)$$

$$= \left(5 \cdot 10 - \frac{100}{10}\right) + 6$$

$$= 50 - 10 + 6 = \boxed{46}$$