Solutions to Final Exam

1. (14 Points) In parts (a)-(e), give a careful definition of the term(s) in bold.

(a) The **kernel** of a group homomorphism $\varphi : G \to H$ is the set $\operatorname{Ker}\varphi = \{a \in G : \varphi(a) = e\}$ where $e \in H$ denotes the identity element of H.

(b) A **normal subgroup** of a group G is a subgroup H that satisfies $aHa^{-1} = H$ for all $a \in G$.

(c) A **field** F is a set with two binary operations $+$ and $\cdot$ such that $(F, +)$ is an abelian group, $(F^\times, \cdot)$ is an abelian group (where $F^\times$ is the set F minus the additive identity 0 for $+$) and the distributive law $a(b + c) = ab + ac$ holds for all $a, b, c \in F$.

(d) A **vector space** over a field F is an abelian group $(V, +)$ together with a map $F \times V \to V$ that satisfies $(\alpha\beta)v = \alpha(\beta v)$, $(\alpha + \beta)v = \alpha v + \beta v$, $\alpha(v + w) = \alpha v + \alpha w$ and $1v = v$ for all $v, w \in V$ and all $\alpha, \beta \in F$.

(e) An **eigenvector** for a linear operator $T : V \to V$ is a non-zero vector $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in F$. ■

2. (14 Points) In parts (a)-(e), decide if the given statement is **True** or **False**. If it is true, give a *brief* explanation. If it is false, explain why or give a counter-example.

(a) The index of $\langle \overline{8} \rangle$ in $\mathbb{Z}_{20}$ is 4. This is **True.** Just note that $20/(20, 8) = 5$ so $\overline{8}$ has order 5 and, hence index 4.

(b) If N is a normal subgroup of a group G and G/N is abelian, then G is abelian. This is **False.** Take $G = S_3$ and $N = \langle x \rangle$. Then $G/N \cong \mathbb{Z}_2$ is abelian but G is not abelian.

(c) There is an isomorphism $\varphi : S_5 \to \mathbb{Z}_{120}$. This is **False.** Note that S_{120} is non-abelian and $\mathbb{Z}_{120}$ is abelian since it is cyclic. Therefore these groups are not isomorphic.

(d) The set of real numbers $\mathbb{R}$ is a vector space over the set of rational numbers $\mathbb{Q}$ under the usual addition and multiplication. This is **True.** Since $\mathbb{Q}$ is a subfield of $\mathbb{R}$, $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

(e) $|\operatorname{GL}_3(\mathbb{Z}_5)| = (124)(120)(100)$. This is **True.** This is a special case of the formula for $|\operatorname{GL}_n(\mathbb{F}_p)|$. ■

3. (12 Points) Suppose that $T : \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$ is given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ 0 \end{bmatrix}.$$

Compute the matrix of T with respect to the standard basis in $\mathbb{Z}_5^3$.

Proof. By taking the image of the standard basis vectors e_1, e_2 and e_3, we see that the matrix T_A is

$$T_A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

■

4. (12 Points) (a) Compute the order of $\overline{9}$ in $\mathbb{Z}_{48}$. We have $|\overline{9}| = 48/(48, 9) = 16$.

(b) Compute the index $[\mathbb{Z}_{48} : \langle \overline{9} \rangle]$. Using part (a) and Lagrange's theorem, we have $[\mathbb{Z}_{48} : \langle \overline{9} \rangle] = 48/16 = 3$.

(c) Identify the quotient $\mathbb{Z}_{48}/\langle \overline{9} \rangle$. Using part (b), we know the quotient $\mathbb{Z}_{48}/\langle \overline{9} \rangle$ has order 3 so that, again by Lagrange's theorem, it must be isomorphic to $\mathbb{Z}_3$. ■

5. (12 Points) Show that every group of prime order is cyclic.

Proof. Let p be a prime integer and let G be a group of order p. Since $|G| > 1$, there must exist and element $e \neq a \in G$. By Lagrange's theorem, the order of a is a divisor of $|G| = p$. But $|a| > 1$ so that $|a| = p$ and hence $|a| = p$. ■

6. (12 Points) Let H be the subset of $\mathrm{GL}_2(\mathbb{R})$ defined by

$$H = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, x \neq 0 \right\}.$$

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. Using familiar matrix arithmetic, we compute

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xu & xv + y \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x & -y \\ 0 & 1 \end{pmatrix}.$$

Together, these computations show H is closed under under multiplication and inversion and hence H is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(b) Show that the map $\varphi : H \to \mathbb{R}^\times$ defined by

$$\varphi\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right) = x$$

is a group homomorphism. The computation in (a) clearly shows that φ has the homomorphism property.

(c) Identify the quotient group $H/\operatorname{Ker}\varphi$. Note that the map φ is part (b) is surjective so that the first isomorphism theorem implies that $H/\operatorname{Ker}\varphi$ is isomorphic to $\mathbb{R}^\times$. ■

7. (12 Points) Let $P_3 = P_3(\mathbb{R})$ be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to 3 and let $\frac{d}{dx} : P_3 \to P_3$ be given by

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2.$$

It is well known that $\frac{d}{dx}$ is linear.

(a) Compute the matrix of $\frac{d}{dx}$ with respect to the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for P_3.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Compute the characteristic polynomial $p(\lambda)$ for $\frac{d}{dx}$. Since the matrix computed in part (a) is upper triangular, the characteristic polynomial is $\lambda^4 = 0$.

(c) Is $\frac{d}{dx}$ diagonalizable? Explain. The differentiation operator is not diagonalizable. We see that 0 is the only eigenvalue from part (b) so the space of eigenvectors is just the kernel of d/dx. But $d/dx(p) = 0$ iff. p is a constant so that the space of eigenvectors is one dimensional and hence there can be no basis of eigenvectors. ■

8. (12 Points) Let V be a vector space over a field F, and let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis for V. Recall that F is a one dimensional vector space over itself and define

$$L(V, F) = \{T : V \to F \mid T \text{ is a linear transformation}\}.$$

For each $j = 1, \ldots, n$, let $T_j \in L(V, F)$ be defined by

$$T_j(v_k) = \delta_{jk} = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{if } k \neq j. \end{cases}$$

Show that $\mathcal{C} = (T_1, \ldots, T_n)$ is a basis for the vector space $L(V, F)$. Is $L(V, F)$ isomorphic to V? Prove your answer.

Proof. We will show that $\mathcal{C}$ spans $L(V, F)$ and is linearly independent. If $T \in L(V, F)$ is any element and we let $a_j = T(v_j)$ for all $1 \leq j \leq n$, then for any basis vector $v_k \in \mathcal{B}$, we have

$$\left(\sum_{j=1}^{n} a_j T_j \right) (v_k) = \sum_{j=1}^{n} a_j T_j(v_k) = a_k = T(v_k).$$

It follows that $\sum_{j=1}^{n} a_j T_j = T$ and hence $\mathcal{C}$ spans $L(V, F)$. Now, if $\sum_{j=1}^{n} a_j T_j = 0$, then for all $1 \leq k \leq n$, we have

$$0 = \left(\sum_{j=1}^{n} a_j T_j \right) (v_k) = \sum_{j=1}^{n} a_j T_j(v_k) = a_k$$

which shows that $\mathcal{C}$ is linearly independent and hence a basis. Finally, we see from above that $L(V, F)$ is isomorphic to V since we have shown that $\dim_F(L(V, F)) = n = \dim_F(V)$, and any two F-vector spaces of the same dimension are isomorphic. ∎