Project Summary

SAGE is an open source general purpose mathematical software system that has developed explosively within the last five years. SAGE-COMBINAT is a subproject whose mission is "to improve SAGE as an extensible toolbox for computer exploration in (algebraic) combinatorics, and foster code sharing between researchers in this area". Among the proposers, Stein is founder and lead developer of SAGE while Bump, Musiker, and Schilling are strong contributers to SAGE-COMBINAT. Hivert and Thiéry (Paris-Sud, Orsay), founders and lead developers of SAGE-COMBINAT, are both strongly affiliated with this project.

The project will develop SAGE-COMBINAT in areas relevant to the ongoing research of the participants, including symmetric functions, crystals, rigged configurations and combinatorial R-matrices, affine Weyl groups and Hecke algebras, cluster algebras, posets, together with relevant underlying infrastructure. The project will include three Sage Days workshops, and will be affiliated with a third scheduled workshop at ICERM. Such workshops always include a strong outreach component and have been a potent tool for connecting researchers and recruiting SAGE users and developers. The grant will also fund a dedicated software development and computation server for SAGE-COMBINAT, to be hosted in the SAGE computation farm in Seattle. Emphasis will be placed on the development of thematic tutorials that will make the code accessible to new users. The proposal will also fund graduate student RA support, curriculum development, and other mentoring.

Intellectual Merits: There is a long tradition of software packages for algebraic combinatorics. These have been crucial in the development of combinatorics since the 1960s. The originality of the SAGE-COMBINAT project lies in successfully addressing the following simultaneous objectives. It offers a wide variety of interoperable and extensible tools, integrated in a general purpose mathematical software, as needed for daily computer exploration in algebraic combinatorics; it is developed by a community of researchers spread around the world and across institutions; and it is open source and depends only on open source software. Developing SAGE-COMBINAT has required both mathematical and computer science expertise. For example, great emphasis is placed in high-level programming techniques (object orientation and polymorphism, iterators, functional programming) to obtain concise, expressive, general, and easily maintained code.

Broader Impact: The combinatorics emphasized in this proposal has applications to diverse areas such as mathematical physics, representation theory, algebraic geometry, and number theory. For example, Hecke algebras arose originally in number theory but have applications in all these fields. The tools in SAGE-COMBINAT will therefore be of use to researchers in these areas. The investigators are all active researchers well cognizant of these connections and SAGE-COMBINAT is developed in the context of actual mathematical research. Due to the rigorous patch review process of SAGE, every tool that is added is scrutinized for appropriate generality and general applicability, resulting in a flexible and maximally useful system. This has already led to cross-fertilization between various areas in mathematics and computer science, and we expect this to continue to be the case.

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Appendix Items:

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Project Description

1 Results of Prior NSF Support

We will report on two FRG grants and a regular grant.

- DMS-0652641 FRG: Collaborative Research: Affine Schubert Calculus: Combinatorial, geometric, physical, and computational aspects.
- DMS-0652817 FRG: Collaborative Research: Combinatorial representation theory, multiple Dirichlet series and moments of L-functions.
- DMS-1067183: Cluster Algebras, Critical Groups, and Tropical Curves

These grants had disjoint membership. Schilling was a PI on the first grant, which also gave support to proposer Thiéry. Bump was a PI on the second grant and Musiker was the PI on the third grant. All grants had this in common: that the research they sponsored made use of Sage, and that some of the code developed is of sufficiently general interest that it was (or will be) submitted to the Sage review process and eventually merged in Sage. Beyond this common interest in Sage as a tool, there are mathematical links between these grants, with commonality in Kashiwara crystals and matters related to the Yang-Baxter equation.

In addition, Sage-Combinat was funded by several French grants ANR-06-BLAN-0380 (2006-2008), PEPS CNRS/INST2I (2009-2010), PEPS CNRS/ST2I (2010) to which co-proposers Hivert and Thiéry were associated.

1.1 Sage

Sage (`http://www.sagemath.org/`) is a completely open source general purpose mathematical software system, which appeared under the leadership of William Stein (University of Washington) and has developed explosively within the last five years. It is similar to Maple, MuPAD, Mathematica, Magma, and up to some point Matlab, and is based on the popular Python programming language. Sage has gained strong momentum in the mathematics community far beyond its initial focus in number theory, in particular in the field of combinatorics.

Development of Sage has been supported by grants from the NSF, most recently DMS-1015114. NSF funded a COMPMATH proposal by Stein in 2007 (DMS-0713225), another COMPMATH proposal in 2011 (DMS-1015114) for Sage Days workshops, and also the purchase of about $120K in hardware via the SCREMS program (DMS-0821725). The NSF FRG grant DMS-0757627 has also indirectly provided substantial support for the development of Sage in number theory. Sage Days 1 was funded by DMS-0555776.

1.2 Code resulting from research on Affine Schubert Calculus grant

The focus of the first FRG concerns the development of a vast extension of Schubert calculus to affine Grassmannians and affine flag varieties, called “affine Schubert calculus”. Classical Schubert calculus, a branch of enumerative algebraic geometry concerned with counting subspaces satisfying certain intersection conditions, is the outcome of the solution to Hilbert's Fifteenth problem. In the modern formulation, Schubert calculus is usually interpreted in cohomology theories of homogeneous spaces, most notably flag varieties. The new approach to affine Schubert calculus was made possible by the discovery of certain explicitly defined symmetric functions called k-Schur functions, which arose in the study of the seemingly unrelated Macdonald theory. The novel combinatorics of k-Schur functions and related objects lent itself to extensive computational experimentation.

The original platform chosen for the development and dissemination of the code resulting from the project was MuPAD-Combinat, founded by Hivert and Thiéry in 2000. In February 2008, all proposers (Bump, Musiker, Schilling, Stein, Thiéry) attended the pivotal Sage Days 7 at IPAM at UCLA. At that meeting, Thiéry and Schilling began porting part of the crystal code from MuPAD to Sage. Due to its many advantages over MuPAD, in particular since Sage is completely open source and has the support of a much bigger community of developers and researchers, it was decided later in 2008 to migrate to Sage under the name Sage-Combinat. Since then, many of the publications associated to the project have used and resulted in contributions to Sage.

For example, in [84, 28, 29] the existence and explicit combinatorial models for Kirillov–Reshetikhin crystals were proven and constructed. Kirillov–Reshetikhin crystals are affine finite-dimensional crystals, which play an important role in lattice models in statistical mechanics and in the construction of highest weight affine crystals using the Kyoto path model. Schilling added support for Kirillov–Reshetikhin crystals to the general framework for crystal bases in Sage. Together with Brant Jones [53], she also considered exceptional types which led to an implementation of fundamental crystals of type E_6 and E_7 as well as their affine counterparts. As a consequence, Jones implemented alcove paths for finite types. All of this code has been integrated into the main distribution of Sage. In [3], Bandlow, Schilling, and Thiéry studied affine crystals in type A coming from promotion operators. This was based on heavy computer exploration, and required the implementation in MuPAD (later ported to Sage) of code for computing isomorphisms of (affine) crystals.

In [4] Bandlow, Schilling and Zabrocki obtained a Murnaghan–Nakayama rule for k-Schur functions. Heavy experiments using Sage gave rise to a simplified definition of a ribbon in this setting. K-theoretic analogues of k-Schur functions were defined in [69], which were implemented in Sage by Schilling to produce tables in the appendix of the paper. A more refined implementation of these functions was done by Thiéry which is almost ready for integration into Sage. Schilling's graduate student Steve Pon worked on analogues of affine Stanley symmetric functions and k-Schur functions for type B and D, generalizing [68] in type C. Pon, Thiéry and Schilling implemented affine Stanley symmetric functions for general type which is now available in main Sage.

In research related to Coxeter groups and 0-Hecke algebras, Hivert, Schilling, and Thiéry, in part with Denton, studied the representation theory of finite-dimensional (monoid) algebras [50, 51, 22]. In particular, they showed that, for an important class of finite monoids, the representation theory is completely combinatorial. Besides providing efficient algorithms to compute the representation theory, this gives a way to naturally associate combinatorial structures, like lattices, to a monoid. This research is still in progress, with ongoing generalizations to larger classes of monoids. To support this research Hivert, Thiéry, and others implemented tools to compute with finite monoids and their representations, including an interface with Semigroupe by Jean-Éric Pin. They also improved other features around root systems, 0-Hecke monoids, Bruhat order, posets, and calculations with vector spaces.

1.3 General infrastructure

In addition to the many research-driven projects that led to implementations integrated into the main branch of Sage, the FRG on affine Schubert calculus was also instrumental with respect to long term investments in terms of software infrastructure. For example, Thiéry (with the help of many) ported to Sage the concept of Categories from Axiom (and followers like MuPAD), blending it with that of Parent and Elements from Magma, and extending it further. The end result is a translation of the mathematical concept (category of groups, of vector spaces, ...) into a concrete software engineering design pattern for promoting generic code, systematic tests, and consistent interfaces, while using the standard Python object oriented and class hierarchy mechanism. This change, representing 30k lines of code, was a major overhaul of the core of the Sage library, and was crucial to the concise and expressive implementation of most of Sage-Combinat features.

1.4 Code resulting from multiple Dirichlet series grant

The second FRG grant supported, among other things, work by Bump and his collaborators on p-parts of multiple Dirichlet series that are not Euler products, but whose coefficients exhibit a twisted multiplicativity. The p-parts of these multiple Dirichlet series may be identified with Whittaker functions on metaplectic groups over p-adic fields. Such a group is a central extension of a reductive algebraic group like $\mathrm{GL}_{r+1}(\mathbb{Q}_p)$ by the n-th roots of unity μ_n for some n dividing $q-1$ where q is the order of the residue field. If $n=1$, the Casselman-Shalika formula shows that these are characters of irreducible representations of the L-group, which is a complex Lie group. In this case, they are given by the Weyl character formula. So, in the general case, these p-parts are a kind of generalization of Weyl characters. They might be described as being like characters in which the weights have been replaced by products of Gauss sums.

These Whittaker functions have a rich combinatorial structure that has been partly exposed by Brubaker, Bump, Friedberg, Chinta, Gunnells and McNamara, among others, in work that was largely supported by the mentioned FRG. They may be expressed as sums over crystals, as partition functions of solvable lattice models, or as ratios of sums over the Weyl group. Each of these descriptions may be regarded as a kind of generalization of the Weyl character formula. In this work Sage was instrumental in discovering basic principles. References [7, 8, 9, 11, 12] are all investigations of this problem that used Sage.

Existing code for working with Kashiwara crystals was then available from Stembridge; however this code, written in Maple, had been broken by Maple upgrades. Luckily, in the spring of 2008, Schilling, Thiéry, and Mike Hansen were starting to migrate MuPAD-Combinat to Sage, with a focus on its crystal features. Brubaker and Bump used the crystal code for many experiments. Bump was able to contribute back to Sage, helping to write the spin crystals needed for Cartan types B and D, and helping to finish the root system code that had been started by Schilling, Thiéry and Hansen by implementing exceptional and reducible Cartan types. Moreover, Bump helped write the code for Weyl Groups, which is a wrapper around the GAP code. All of these contributions benefited from input and sometimes refactoring by Thiéry.

Bump contributed an important module to Sage, the WeylCharacter code [110, 111, 112, 114], which is the context in which most typical Lie group computations can be performed. One may compute the character of an irreducible finite-dimensional representation of any Lie group as an element of a ring. This gives access to its weight multiplicities, Clebsch-Gordan decompositions (that is, decomposition of tensor products into irreducibles), and branching rules. This code has been merged into Sage, and additional documentation [113], which shows by examples how to use Sage for Lie theory, is now standard documentation.

Bump and Nakasuji used Hecke algebras to compute intertwining integrals and generalizations of the formula of Gindikin and Karpelevich in [15]. Brubaker, Bump and Licata [13] gave a different implementation based on formulas of Casselman and Reeder. Both investigations required extensions to Sage. These included the Bruhat order for Weyl groups, Iwahori-Hecke algebras, and Kazhdan-Lusztig polynomials. These extensions were of sufficient general importance that they were eventually merged into Sage [115, 116, 117].

Sage played a role in the investigation of representations of Whittaker functions and characters of irreducible representations of Lie groups by solvable lattice models. See Brubaker, Bump and Friedberg [10], Brubaker, Bump, Chinta, Friedberg and Gunnells [12]. This investigation, particularly the investigation of the Yang-Baxter equation to show that the partition functions are symmetric, used Sage extensively but has not yet resulted in any code being merged back into Sage. Sage has been key in their work in progress of Brubaker, Bump and Licata [13] on Iwahori Whittaker functions.

1.5 Code resulting from Cluster Algebra grant

In comparison with the previous two mentioned grants, grant DMS-1067183 is a standard grant as opposed to an FRG and has the specific mathematical topics of cluster algebras, sandpile groups, and tropical geometry as its focus (topics which all have interconnections between them). Part of this grant involves funds for undergraduate research utilizing computational software and other methods for finding patterns that motivate new conjectures and inspire further mathematical research. There has been a long tradition of such undergraduate research in these topics, such as Jim Propp's NSF funded programs REACH [89] and SSL [90]. This summer at Minnesota, with co-mentors Pylyavskyy and Reiner, Musiker is mentoring an REU of eleven undergraduate students on a variety of combinatorial topics. For several of these topics, for example, sandpile groups and cluster algebras, computer experimentation in Sage has been a valuable tool.

Since that proposal, Musiker and Stump both attended Sage Days 20.5 at the Fields Institute in May of 2010, alongside Hivert, Schilling and Thiéry, and through this meeting, development of code in Sage for cluster algebra computations has taken off. Working with Christian Stump and with this grant support in Fall 2010, a public version of the first iteration of a cluster algebra for Sage was completed in January of 2011. The code and relevant theoretical background is presented in an accompanying compendium [83]. This code is now being migrated into Sage using the Sage-Combinat patch system (see Section 3.3). Some features are:

- matrix and quiver mutations (corresponding to the seed of a cluster algebra).
- computing Laurent polynomial expansions of cluster variables.
- numerous constructors for creating a seed for a cluster algebra, including cluster algebras of finite type, which are classified by the same Cartan-Killing classification of Lie groups.
- procedures for generating the entire mutation-class of cluster algebras/matrices/quivers which are mutation-equivalent to a given initial seed. Due to the speed of the algorithms in Sage, systematic studies of seeds of finite or infinite mutation types are both accessible.
- given an arbitrary exchange matrix or quiver, tests to see whether such seed data gives a cluster algebra of finite type; and if so, gives the type (in many cases such as finite or affine type) corresponding to the above classification and those from work of Felikson-Shapiro-Tumarkin

[30, 31]. This Sage package is the first computational software that recognizes cluster algebras of finite type from seed data in this way, utilizing cutting edge results regarding type testing.

Musiker plans to demonstrate this software as a lecturer/organizer of one of MSRI's Summer Workshops for Graduate Students in August 2011. He has also been utilizing this code with undergraduates as part of a research project, as mentioned above. In particular, an REU project already underway utilizes this software to discover patterns in Laurent polynomials (cluster variables) arising from certain periodic quivers. These Laurent polynomials are elements of cluster algebras of infinite mutation type, but there is still enough structure in these examples so that combinatorial patterns are possible to uncover. For example, such periodic quivers include those arising from the Gale-Robinson and Somos sequences [37]. A number of future REU projects will also use this software, one possibility being a further study of cluster algebras associated to Q-systems and T-systems, which come from mathematical physics and statistical mechanics [24].

2 Symbiotic relationship

The originality of the Sage-Combinat project lies in successfully addressing the following simultaneous objectives. It offers a wide variety of interoperable and extensible tools, integrated in a general purpose mathematical software, as needed for daily computer exploration in algebraic combinatorics; it is developed by a community of researchers spread around the world and across institutions; and it is open source and depends only on open source software.

Reaching this scale is a true challenge, as there is a simultaneous need for mathematical and algorithmic expertise and for strong computer science experience, in particular concerning the design and the development model. For example, Sage-Combinat puts great emphasis in high-level programming techniques (object orientation and polymorphism, iterators, functional programming) to obtain concise, expressive, and easy to maintain code.

The Sage-Combinat project (with its ancestor MuPAD-Combinat) is by nature research driven and already led to more than 70 publications [98, 107]. Its development model, which fosters code sharing, has already had tremendous success by stimulating research interactions across disciplines and across continents as should be evident from the above testimony. For example, the code in Sage for Kashiwara crystals written by Schilling and Thiéry has been used by number theorists in their study of multiple Dirichlet series. Similarly, work on Iwahori intertwining operators by Bump and Nakasuji led to inclusion of code for the Bruhat order and Iwahori-Hecke algebras in Sage, code that has already been used by other mathematicians. This is in part because the rigorous review it was given during the Sage development process led to revision of the code with the proper generality and relationship with existing code.

Due to its symbiotic relationship between research pursued by individuals for their own work and the development of generally useful code, Sage-Combinat was also chosen as the platform for a semester-long ICERM program in 2013 [1], which seeks to explore connections between multiple Dirichlet series, crystal bases, and Schubert calculus via computational tools.

3 Software Development

An important aim of this project is to facilitate research in algebraic combinatorics, representation theory, and number theory by intensive computer exploration and at the same time develop

[1] http://icerm.brown.edu/sp-s13/

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code that can be used by a general audience through inclusion in the main branch of Sage:

- **Computer exploration**. This is a crucial research tool for all the participants, through the creation and analysis of large sets of data, and the testing of conjectures. The computer exploration underlying this research project will rely on intensive computations and advanced features, models, and algorithms, many of which have never been implemented or even are yet to be invented.
- **Mutualization of our software development efforts**. Focusing on a single platform will allow us to easily share code, experiments, large scale databases of results, etc.
- **User interface testbed**. The intensive use of Sage-Combinat by non-developers in a collaborative research situation will give the Sage-Combinat developers precious feedback on the user interface.
- **Outreach**. Sage-Combinat takes the form of a collection of experimental extensions (patches), developed jointly by a community of researchers. These extensions are intended to have a short life cycle, and to be merged into Sage as soon as they are mature enough. Hence, in a matter of months, most new features are made available to all Sage users.

  Experience further shows that, thanks to the high level programming framework (object orientation, polymorphism, categories), most of the required code can be written in a generic way and reused in widely different contexts.

  This prompt integration of our developments into Sage will make them easily accessible as a robust toolbox for other researchers in mathematics, physics, and computer science.
- **Teaching**. We plan to integrate Sage into the curriculum of the courses at our universities. Many undergraduate and graduate courses in number theory and combinatorics already involve computational components. With dwindling resources a free and open-source alternative to Mathematica and MatLab is most welcome. Since Sage is based on Python, which is a widely used language in computer science, this also gives the students valuable education. Inspired students can furthermore choose to contribute to Sage themselves, which is impossible with propriety systems!

3.1 The Sage and Sage-Combinat projects

Sage [96] is a free open-source mathematics software licensed under GPL. It combines the power of many open-source packages (GAP [38], Linbox, Singular [42], Symmetrica, etc.) into a common Python-based interface.

The mission of Sage-Combinat [97] itself is to improve Sage as an extensible toolbox for computer exploration in combinatorics, and to foster code sharing between researchers in this area. Sage-Combinat started in 2000, under the leadership of Hivert and Thiéry, as an open source library MuPAD-Combinat [49] for the computer algebra system MuPAD [80]. It took its roots in the projects ACE [108], μ-EC [91], PerMuVAR [106], and progressively integrated CS [19], Symmetrica [58], Nauty [77], and lrcalc. In June 2008, it was decided to migrate the project to Sage, then mature enough to benefit from the enhanced dissemination offered by a fully open source platform. The large overhead of this migration (100k+ lines of code) was compensated by the almost doubling of the community worldwide, by joining our efforts with the combinatorics on words project Sage-words, and as described in the section on prior NSF support by a very stimulating cross-fertilization with number theorists. Sage-Combinat and its ancestor MuPAD-Combinat have already been an invaluable research tool for suggesting and testing conjectures, playing an essential role in over 70 publications (see [98, 107]).

3.2 Related software

There is a long tradition of software packages for algebraic combinatorics [99]. For example, there already exist libraries for Maple [17] implementing features like symmetric functions, root systems, and some structures in representation theory, like ACE [108] (Institut Gaspard Monge) and SF,Coxeter/Weyl [104] (John Stembridge); they were used, for example, by Lenart to implement the alcove model for crystals [72]. However, these packages are hindered by the lack of some natural computer science paradigms (object oriented programming, functional closures, etc.) in Maple. And of course, Maple is not open source.

The software GAP [38], with its Chevie package [40], and Lie [73], although too specialized for our needs, are great sources of inspiration in their respective areas. In fact, GAP4 comes bundled with Sage; however the interface needs further development to better expose GAP4 features. Another technical issue is that Chevie is based on GAP3. Improving this situation is the topic of an ongoing collaboration with GAP and Chevie developers, which started at Sage Days 20 in February 2010, and was further pursued during a joint workshop organized in June 2010 in Orsay[2].

Another important package is Coxeter3, by the late Fokko du Cloux [27]. This package is blazingly fast at computing Kazhdan-Lusztig polynomials and related algorithms. Kazhdan-Lusztig polynomials were implemented in Sage (by Bump) but the algorithms in Coxeter3 are faster. An experimental Sage interface for Coxeter3 exists by Mike Hansen. Anders Buch's very fast *C* package lrcalc, which computes Littlewood-Richardson and fusion coefficients, is in the process of being integrated into Sage by efforts of Buch, Hansen, Schilling, and Thiéry [120].

3.3 Development model

The collaborative nature of the project requires advanced software development tools (ticket server, distributed version control), as well as heavy computational resources (multi-platform compilation, regression testing, benchmarking). So far, we have been using a virtual server courtesy of the Sage team at the University of Washington (`combinat.sagemath.org`). Scaling further will require its replacement by a modern machine, which the PI Stein will administer for this project as part of the Sage computation farm. It will be accessible remotely to all participants, through ssh or the Sage notebook interface. This server will also be used for time or memory demanding calculations. Unused resources will be made available to the Sage-Combinat and Sage community at large. See the budget justification for UW for details.

Most of the code we develop will be shared among the participants using the Sage-Combinat patch server, and thus instantly publicly available. It will be progressively integrated into the official version of Sage, as and when it reaches maturity, following the standard Sage review process. Technically, a patch is an incremental change to the Sage code base such as the addition of a new feature or a bug fix. The purpose of the Sage-Combinat patch server is to ease collaborative development and in particular the review process, to make features quickly available for fostering user feedback (release early, release often), and to anticipate the merging of the patches into Sage in order to detect potential conflicts as early as possible (preintegration). Some pieces which are too specific or experimental for eventual inclusion into Sage will

[2] `http://wiki.sagemath.org/combinat/SageCombinatChevieWorkshopOrsay2010`

be shared through a private mercurial server. This private server together with a wiki will also be used to share log books of experiments, as well as to write articles, reports, etc. Meanwhile data tables generated by our software will be published on the web on our public wiki. Pending sufficient interest, they will be further distributed as optional SAGE packages.

All proposers have extensive experience with the development of SAGE code. In particular, Thiéry is very experienced in managing and animating team software development in the context of a large research project. He together with Hivert will act as consultants on the software design and algorithmic aspects, coordinate the development, and implement the most technical core functionalities (beside what they will need for their own research).

4 Target features

4.1 Symmetric functions

Symmetric functions are power series of bounded degree in variables $x_1, x_2, \ldots$ which are invariant under all permutations of the variables. An important basis for the space of symmetric functions are the Schur functions. These have applications in geometry (where they can be interpreted as representing cohomology classes for Grassmannian varieties) and representation theory (where they represent the characters of the symmetric and general linear groups). A rich combinatorial theory of tableaux allows efficient calculation with Schur functions.

This theory is, for the most part, well-implemented in SAGE, due to the inclusion of the package SYMMETRICA. An important exception is that the Hopf algebra structure for the ring of symmetric functions is not currently accessible. Implementing tools for computing with this structure is critical for the seamless integration of the existing symmetric function code with the quasisymmetric and non-commutative symmetric function code which we will be developing.

There are many more modern families of polynomials closely related to symmetric functions which do not currently have implementations in SAGE. Examples include quasisymmetric and non-commutative symmetric functions, Schubert and Grothendieck functions, the polynomials of Lascoux, Leclerc and Thibon, and Demazure atoms. Each family can be described with combinatorics similar to the combinatorics of tableaux. Hivert is planning a framework for rapid prototyping of such combinatorial objects.

Another family of symmetric functions for which we intend to improve support is the family of k-Schur functions. This family was introduced by Lascoux, Lapointe and Morse in a study of Macdonald polynomials [70]. k-Schur functions are indexed by a positive integer k and a partition whose largest part does not exceed k. They are Schur-positive symmetric functions that form a basis for the subring $\mathbb{Z}[h_1, \ldots, h_k]$ of symmetric functions, where the h_i are the complete homogeneous functions. k-Schur functions are deformations of Schur functions, in the sense that for large enough k, they are equal to Schur functions. They have connections to many other areas of mathematics. In particular, they form a Schubert basis for the affine (loop) homology of the Grassmannian [65, 67]. Furthermore, the structure coefficients of the ring of k-Schur functions are known to contain the fusion coefficients for the Wess-Zumino-Witten conformal field theories associated to $\widehat{su}(\ell)$ [71]. The dual basis to the space of k-Schur functions (and the quotient space of the ring of symmetric functions which they span) is also interesting. The basis is a natural “affine” generalization of the Stanley symmetric functions [64].

Limited support for k-Schur functions is currently available in SAGE. In particular, some support for the dual k-Schur functions is available via the affine Stanley symmetric function code, introduced by Pon, Schilling, and Thiéry. We will extend this support by defining the space in which k-Schur functions live, defining the dual space, and implementing the dual basis. A prototype was implemented by Bandlow, Schilling, and Thiéry at Sage Days 20.5 and Sage Days 30. The combinatorics of k-Schur functions and their dual-basis are governed by a subset of partitions known as cores, and fillings of these objects which are known as strong/weak tableaux. We will implement support for these objects as well.

There are other families of symmetric functions that should be implemented. Macdonald polynomials are extremely important symmetric polynomials involving two deformation parameters. In Sage they are currently implemented for Cartan type A only. It is proposed to extend this to other Cartan types. This project will involve Bump and Schilling. McNamara [78] and Bump, McNamara and Nakasuji [14] found lattice model representations of factorial Schur functions, with R-matrix interpretations. It would be highly desirable to find similar lattice model representations for k-Schur functions. Another important family of symmetric functions are induced by the action of the symmetric group on rational functions via the birational R-matrix. The polynomial invariants of this action were recently studied by Lam and Pylyavskyy [66] under the name loop symmetric functions. They play a role in the theory of discrete dynamical systems as in the works of Takahashi and Satsuma [105], and geometric crystals as in the works of Kashiwara, Nakashima, and Okado [56]. Study of symmetric functions is computation intensive and would benefit from software code dedicated to that ring.

Many bases for symmetric functions are obtained by symmetrizing bases for multivariate polynomial rings. Important examples include the non-symmetric Macdonald polynomials (which symmetrize to Macdonald polynomials) and the Demazure atoms (which symmetrize to Schur functions). By work of Ion [52], non-symmetric Macdonald polynomials specialized at $t = 0$ are nothing else but Demazure characters. Using the methods described in Section 4.3 they can be calculated using the energy function on crystals. Alternatively, an efficient implementation of the combinatorial definition of Haglund, Haiman, and Loehr [43] of the non-symmetric Macdonald polynomials can be used for the calculation of the energy function.

A prototype including several such bases, and the associated combinatorial objects, is being developed by Vivianne Pons in Marne-la-Vallée, under the supervision of Alain Lascoux, in collaboration with Nicolas Borie and Adrien Boussicault. Support for Demazure characters is also already available using connections to Demazure crystals. We will further expand this prototype to a full implementation in collaboration with the team in Marne-la-Vallée and Orsay.

4.2 Combinatorial Hopf algebras, operads, and applications

Quasisymmetric functions are power series of bounded degree in variables $x_1, x_2, \ldots$ which are invariant under order-preserving shifts of the variables. This is to say that $x_1^{m_1} \cdots x_k^{m_k}$ and $x_{i_1}^{m_1} \cdots x_{i_k}^{m_k}$ have the same coefficient for any strictly increasing sequence of positive integers $i_1 < \cdots < i_k$. These functions were defined in the 80's by Ira Gessel, who developed many of their basic properties, and applied them to permutations enumeration [41].

The subring QSym of quasisymmetric functions is now a fundamental tool for researchers in algebraic combinatorics. It naturally contains the ring of symmetric functions, and is the dual Hopf algebra to the ring of noncommutative symmetric functions (known as NCSF) [79]. Quasisymmetric expansions have been used to prove the Schur-positivity of Macdonald polynomials [1] and are used by Assaf and Billey to prove Schur-positivity of k-Schur functions, relying heavily on computer checks. Quasisymmetric functions can be considered as representatives of the characters for the representation theory of the 0-Hecke algebra [25]. Recently, a new basis for the space of quasisymmetric functions was defined which naturally refines the Schur basis for symmetric functions [44]. This basis is described by objects known as composition-tableaux.

Further generalizations of symmetric functions, usually nicknamed combinatorial Hopf algebras and involving various kinds of combinatorial objects such as permutations, parking function, trees, or graphs, have also received great interest in the past few years. They indeed have a broad range of applications from algorithm analysis in computer science (as very refined generating series) [48] to resummation of diverging series through mould calculus [26, 6] in mathematics and renormalization in quantum fields theories [18]. In this rapidly growing field, various new types of algebras (prelie, dendriform, brace algebras) have also appeared using the operad formalism.

Limited support for QSym and NCSF is currently available in Sage due to efforts of Thiéry and Hivert. This functionality will be extended by including more bases and algebras (Loday-Ronco, Grosmann-Larson, Connes-Kreimer). A prototype for the implementation of operads is available thanks to Hivert and Chapoton. We will expand it further in order to compute in various kinds of algebras. Furthermore, support for objects such as trees and composition-tableaux will be provided. Part of this work involves porting existing code from MuPAD-Combinat to Sage. Another part (particularly, that relating to composition tableaux) involves the development of novel code, as no public implementation of these algorithms currently exists.

4.3 Rigged configurations and the combinatorial R-matrix

Crystal graphs [55] consist of a non-empty set B on which the Kashiwara operators e_i and f_i for i in some index set I act. The Kashiwara operators can be thought of as the Chevalley operators of the underlying (Kac–Moody) Lie algebra in the crystal limit of the quantum parameter $q \to 0$. For affine crystals, that is, crystals corresponding to an affine Kac–Moody algebra there exists an important class of finite-dimensional crystals, called Kirillov–Reshetikhin crystals [47, 46]. Since in this setting tensor products are irreducible, there is a notion of a combinatorial R-matrix [54] which describes the affine crystal isomorphism $R : B_1 \otimes B_2 \to B_2 \otimes B_1$.

The R-matrix plays a fundamental role in Yang-Baxter equations, statistical mechanical lattice models, their energy functions [86], and the inverse scattering method for periodic box ball systems [62, 63, 100, 101, 102]. It is highly desirable to have an efficient algorithm to calculate the combinatorial R-matrix in Sage. This topic has potential applications in the theory of Whittaker functions; see [12].

Currently, there exists an algorithm for R based on the fact that it describes a crystal morphism. However, for large crystals, this algorithm is extremely inefficient. By work of Kirillov, Schilling, and Shimozono [61] and later extensions [85, 23], there is a bijection from tensor products of crystals to rigged configurations. Rigged configurations are combinatorial objects that first arose in the setting of the Bethe Ansatz [59, 60] and reflect the quasi-particle structure of the underlying lattice model. It is known [61] that the R-matrix intertwines with the identity map under the bijection between crystals and rigged configurations. An implementation of this bijection would hence also yield a much more efficient algorithm for the combinatorial R-matrix for large crystals, as it scales with the size of the tableaux describing the crystal elements rather than the size of the crystal itself. Schilling together with graduate student Travis Scrimshaw are planning to work on an implementation of these algorithms.

4.4 Affine Weyl groups and Hecke algebras

Affine Weyl groups and their Hecke algebras are two topics of remarkably broad applicability. They arose first in the work of Iwahori and Matsumoto on generalized Bruhat decompositions

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in p-adic groups. Apart from their historical origin in number theory (where they remain important), affine Weyl groups are a basic feature in string theory and, as the work of Kazhdan and Lusztig has shown, their Hecke algebras appear in a remarkable number of topics such as Verma modules and the geometry of Schubert varieties.

As explained in "Results of Prior NSF Support" affine Weyl Groups and Iwahori-Hecke algebras were implemented in Sage by Bump et al. Let us begin with a classical Cartan type of rank r. The Weyl group W has generators s_i (the simple reflections) with $1 \leq i \leq r$ satisfying $s_i^2 = 1$ and that the braid relations $(s_i s_j)^{m(i,j)} = 1$ where $m(i,j)$ may be read off from the Dynkin diagram. The Iwahori-Hecke algebra has generators T_i that satisfy the same braid relations but in place of $s_i^2 = 1$ one has $T_i^2 = (q-1)T_i + q$, where q is an indeterminate. The affine Weyl group W_{aff} and its Hecke algebra add another generator s_0 to W and another generator T_0 to the Hecke algebra. Both Weyl groups are Coxeter groups, and they, with their Hecke algebras are implemented (in appropriately greater generality than we have described). However there is a larger group, the *extended affine Weyl group* that was constructed by Iwahori and Matsumoto, which is an extension of the affine Weyl group by the fundamental group, a finite abelian group.

The extended Weyl group is not a Coxeter group. It needs to be implemented in Sage. This will have important applications, in particular to affine Demazure characters mentioned in Section 4.1. In particular, Demazure characters associated to translation elements in the extended affine Weyl group correspond to non-symmetric Macdonald polynomials.

Bernstein and Zelevinsky (unpublished but well-known) gave another presentation of the affine Weyl group that is very important. (A published account was given by Lusztig.) As was known to Iwahori and Matsumoto, the affine Weyl group is a semidirect product of the finite Weyl group by the lattice of coroots or (for the extended group) the coweight lattice. The Bernstein-Zelevinsky presentation is an analog of this fact. This presentation as well as particular Hecke algebra elements described by Kazhdan and Lusztig need to be implemented in Sage. Bump, Schilling and Thiéry will be involved in this project.

4.5 Cluster Algebras

Cluster algebras, defined by Fomin and Zelevinsky [35], are certain commutative algebras which are isomorphic to subalgebras of the field of rational functions. Each cluster algebra has a distinguished set of generators called cluster variables; this set is a union of overlapping algebraically independent finite subsets called clusters, which together have the structure of a simplicial complex. The clusters are related to each other by binomial exchange relations. In the past decade, such algebras have been found to be related to a number of other topics such as quiver representations, tropical geometry, canonical bases of semisimple algebraic groups, total positivity, generalized associahedra, Poisson geometry, and Teichmüller theory.

One of the first theorems in theory of cluster algebras is the result by Fomin and Zelevinsky [36] that all cluster variables can be expressed as a Laurent polynomial with integer coefficients. This led to the conjecture that these coefficients are positive, where positivity of the coefficients is significant, as it is conjecturally related to total-positivity properties of dual canonical bases [33, 34, 74, 109]. Nonetheless, this conjecture is still open despite nearly a decade of work by many researchers proving it for certain families of cluster algebras.

Because of their algebraic definition, cluster algebras as a topic are quite amenable to computer exploration. Thus the initial data of a cluster algebra seed can be given as an exchange matrix encoding the associated mutation rules. This has been the inspiration for the cluster algebra library for Sage, developed jointly by Musiker and Stump, as described in Section 1.5.

Musiker has also worked with Schiffler and Williams on the case of cluster algebras arising from surfaces [82]. This research was started during his position as an NSF Postdoctoral Fellow, and has continued through the Spring 2011. Cluster algebras from surfaces are quite an important and large class of cluster algebras so understanding them and showing positivity of the associated generators, i.e. cluster variables, has been quite valuable. In particular, Felikson-Shapiro-Tumarkin [30] have shown that the set of cluster algebras that are skew-symmetric and finite-mutation type (those for which there are only a finite number of reachable exchange patterns no matter how many times one applies exchange relations) is equal to set of cluster algebras arising from surfaces, plus rank two cluster algebras, with *only* 11 other exceptional cases. This classification was only accomplished because of software that the three of them wrote, see `http://www.math.msu.edu/~mshapiro/FiniteMutation.html`, but is a crucial theorem in the theory of cluster algebras of finite mutation type. Using these methods, it has been possible to encode algorithms that take as input the data (such as the exchange matrix) for a cluster algebra, and outputs an answer of whether or not that cluster algebra is finite mutation type. Furthermore, our algorithms in SAGE thus far can classify those of finite mutation type according to the Cartan-Killing classification in many cases, making the methods for the proof of this classification more publicly accessible.

There are numerous enhancements to SAGE that would be quite useful for cluster algebra computations. This work would continue to include Christian Stump, as well as graduate students at the University of Minnesota:

• Include new constructors such as obtaining a cluster seed from an ideal triangulation of a Riemann surface [32]. This would involve interfacing with topologists such as as John Palmieri of University of Washington who have independently expressed interest in procedures for manipulating ideal triangulations such as bistellar flips. This work would also involve improvements to SAGE's graphical interface and graph editor.

• Make the methods for classifying cluster algebras of finite mutation type even sharper. In particular, implement methods for telling to which surface (i.e. what genus, what boundary components, and numbers of marked points on boundaries and interior) a given cluster algebra of finite mutation type is associated, when appropriate. This goal actually motivates interesting theoretical problems in addition to computational ones. It would also be useful to include the classification of the exceptional finite mutation type cases not coming from surfaces, in both the skew-symmetric, and non-skew-symmetric (but skew-symmetrizable) cases.

• algorithms for working with quiver representations such as computing the Auslander-Reiten quiver of the associated path algebra; implementing the algorithms from Derksen-Weyman-Zelevinsky's Quivers with Potentials [20]. It would also be beneficial to provide the user with the ability to change quivers, as in Keller's applet in Javascript [57].

• procedures for quantum cluster algebras (will involve work with Dylan Rupel) and canonical bases of various coordinate rings such as Grassmanians.

• enhancing SAGE's capabilities for working with knots such as Jones polynomials and Kaufmann brackets, which would also be useful for computations associated to quantum or classical cluster algebras arising from surfaces.

4.6 Posets

Posets are ubiquitous in (algebraic) combinatorics (see e.g. Chapter 3 of Richard Stanley's treatise Enumerative Combinatorics [103]), especially when it comes to computations. The original poset library in SAGE was authored by Franco Saliola, during and after his participation to Sage

Days 7. It includes many standard features, like Möbius functions or the generation of all posets of a given size up to an isomorphism. Since then, it received contributions from many, including a recent major refactoring and feature addition by Chapoton, Stump, and Thiéry [119]. The participants to this grant will need, and therefore contribute to the implementation of, further features including tests for Spernerity, P-partition enumeration, quasisymmetric functions associated to a poset, Greene-Kleitman invariant, MacNeille completion, incidence algebras, cd-index of an Eulerian poset, Up-Down operators in differential posets, Stanley-Reisner ring, shellability, homology calculations, and Zeta polynomials. It would also be an improvement to extend SAGE's library of posets with the posets of non-crossing partitions for other Coxeter groups, Cambrian lattices from Nathan Reading [92], Tamari lattices, together with a generator for all (distributive) lattices. Adding support for infinite and lazy posets will prove essential to, e.g., manipulate the Bruhat order for affine Weyl groups.

There are natural connections between the topics of these enhancements and the topics from previous subsections, such as quasisymmetric function methods being then applicable to the function associated to a poset, and bijections between cluster algebra seeds and elements in the Cambrian lattice. Such enhancements would be developed by combinatorics graduate students at University of Minnesota under the supervision of Musiker and Reiner.

4.7 Representation theory of finite-dimensional algebras and semigroups

The last few years have witnessed a fruitful convergence of the algebraic combinatorics and the semigroup theory communities on the topic of representation theory of semigroups. Indeed many algebras of interest in algebraic combinatorics (see e.g. Section 1.2) are in fact semigroup algebras; much recent progress was due to the combination of techniques and computational tools from algebraic combinatorics and from semigroup theory (see e.g. [95, 76, 39, 21, 2, 50, 51, 22]). A strong aspect of this research is the development of efficient algorithms and implementations, which in turn foster further discoveries. Target features already under active development:

- Flexible support for constructing semigroups, algebras, representations, modules, ...
- Computation of Cartan matrices, quivers and relations, radical filtrations, ...
- Computation in character rings.

Even though the focus is on semigroup algebras, using sufficient generality in the implementation will allow for calculating the representation theory of general finite-dimensional algebras.

All of these features interact strongly with the previous projects: we have already mentioned that many Hopf algebras, and at the forefront Sym and QSym, are character rings of group or semigroup algebras; similarly quiver and quiver representations play an important role in cluster algebras. Finally, crystals are semigroup modules in a natural way, and the impact of this fact is yet to be explored. Thus, this project led by Thiéry will require a tight collaboration with all participants and occasional invitees.

4.8 Infrastructure

All described projects depend on further improvements to SAGE's infrastructure, which in turn will have a broad impact on SAGE:

> *"I certainly did not realise when the combinat people joined* SAGE *how useful they and what they do would be for people like me!"* John Cremona, number theorist.

Specific areas include:

- Categories, morphisms, coercions, functorial constructions (Thiéry, Simon King et al.);
- Vector spaces, subspaces, quotients, subalgebras (this point will strongly benefit from the involvement of Stein, who is the author of the original Sage implementation);
- Factories of enumerated sets (infrastructure for enumerating sets of combinatorial objects with various constraints), combinatorial species (enumeration of sets of objects recursively defined), optimization and parallelisation of combinatorial generation (Hivert et al.).

5 Proposed events

5.1 Sage Days

We are planning to hold three Sage Days in the US associated to this grant, though one of these (at ICERM) is not funded by the grant. Musiker is planning to host Sage Days at IMA (Institute for Mathematics and its Applications) at the University of Minnesota during Summer 2012 (see attached letter from Fadil Santosa). During the spring of 2013 a one-semester long program on multiple Dirichlet series and computational algebraic combinatorics has been approved at ICERM. Bump, Schilling, and Thiéry are co-organizers of this program and a lot of the research outlined in this proposal will also be very relevant to the ICERM program. Hivert, Schilling and Thiéry will organize Sage Days at ICERM during this time. In addition to these three workshops, Hivert and Thiéry will organize Sage Days in Paris in summer 2013, as a satellite event of FPSAC, the main yearly international conference in algebraic combinatorics. During the year 2013-2014 Bump and Schilling will organize Sage Days at UC Davis. These conferences will be a similar length, a five day work week, as previous Sage Days, and would have participants ranging from graduate students and postdocs to senior researchers in the area. The goal is for developers to work first-hand with Sage users, and to recruit new users and developers. Working both with the IMA and ICERM would also bring a potentially new audience to Sage with researchers who are especially interested in computational mathematics.

Stanford University, UC Davis and the University of Minnesota may also host smaller meetings of the Sage-Combinat developers as there are good facilities for such meetings. For meetings at the University of Minnesota, Vic Reiner would also be involved in such meetings.

5.2 RA support and curriculum development

At UC Davis several graduate students have already been involved in Sage on various levels. Both Steve Pon and Tom Denton have contributed code that has been merged into Sage. Qiang Wang has used Sage (and before that MuPAD) extensively for his research on the cyclic sieving phenomenon. Tom Denton has also helped organize small group meetings to promote Sage to other students and visitors, in particular Jia Huang from the University of Minnesota. Travis Scrimshaw is currently involved in writing code for rigged configurations.

During the Fall Quarter 2010, Schilling has used William Stein's book on number theory in addition to the usual textbook by Rosen for the upper division undergraduate class MAT115A (see `http://www.math.ucdavis.edu/~anne/FQ2010/mat115A.html`), and has integrated Sage examples and exercises into the class. It is planned to integrate Sage also into the curriculum of other classes, such as MAT145 Combinatorics, MAT165 Mathematics and Computers, and possibly the lower division undergraduate classes on linear algebra MAT22A and MAT67.

This grant will be used at the University of Minnesota for Musiker and Reiner to contribute to the Sage-Combinat project, and in particular further develop software for the aforementioned topics of posets, quasisymmetric functions (as well as loop symmetric functions with Pylyavskyy), and cluster algebras, while providing graduate students with more opportunities to work with and work on Sage. The University of Minnesota has several active graduate students in combinatorics who are both interested in Sage, and studying the relevant mathematical topics: Adil Ali, Alex Csar, Kevin Dilks, Alex Garver, Jia Huang, Thomas McConville, Alex Miller, Nathan Williams, and we anticipate having a steady state of more such graduate students in the near future. Musiker recently taught a topics course for graduate students in spring 2011 which introduced the students listed above to the topics of cluster algebras and quiver representations.

The graduate student RA support asked for in this grant will expose beginning researchers to the combinatorial aspects of Sage; this can be a powerful tool for students trying to master the computationally intricate theory common in combinatorics. Furthermore, it will provide them with an opportunity to develop (with guidance) code implementing novel algorithms for research-level mathematics. The capabilities of and possible enhancements to the Sage-Combinat package have been described previously in this grant. In particular, many combinatorics students at the University of Minnesota, UC Davis, and Stanford are knowledgable and interested in topics such as quasisymmetric functions, posets, Coxeter groups, Kazhdan-Lusztig polynomials, crystal bases, and root systems.

6 Thematic Tutorials

Good documentation is essential for software in general, and specifically for Sage and Sage-Combinat. Luckily, Sage's documentation system has excellent support for mathematics, and it includes a comprehensive reference manual, thanks to its strict review process which enforces documentation on all new code.

Yet, this is not sufficient; an emerging trend is the development of a collection of *thematic tutorials* included in Sage's documentation. Each such tutorial aims at giving an overview of the features in Sage on a given theme, together with relevant mathematical background. Those tutorials are important for making sure that mathematicians are fully aware of the relevant tools available within Sage. As a corollary, they are important for outreach to new users and developers. Finally, they help maintain the cohesiveness of Sage by disseminating information within the existing base of users and developers.

A typical such tutorial relevant to this grant is the thematic tutorial on Lie group methods and related combinatorics, by Bump and Schilling ([113]), now part of the standard documentation. Planned extensions include a chapter on k-Schur functions (Schilling) and one on Coxeter groups. A second example is the chapter on combinatorics (by Thiéry) in [121]. Other tutorials in progress may be found in the Sage-Combinat mercurial queue: "Using Free Modules," (Bandlow) "Implementing Algebraic Structures" (Bandlow-Thiéry), and "Objects and Classes" (Hivert). Some RA support money will be dedicated to the development of such thematic tutorials. Indeed, under appropriate supervision, writing such a tutorial is a great way for a student to learn a theory together with its implementation in Sage.

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`http://www.emis.de/journals/SLC/divers/software.html`

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[112] `http://trac.sagemath.org/sage_trac/ticket/5794`

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[113] This thematic tutorial *Lie methods and related combinatorics* is available on-line at http://www.sagemath.org/doc/thematic_tutorials/lie.html

[114] http://trac.sagemath.org/sage_trac/ticket/7992

[115] http://trac.sagemath.org/sage_trac/ticket/7753

[116] http://trac.sagemath.org/sage_trac/ticket/7751

[117] http://trac.sagemath.org/sage_trac/ticket/7922

[118] http://trac.sagemath.org/sage_trac/ticket/10167

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Section E: Biographical Sketch – Anne Schilling

a. Professional Preparation

Undergraduate Institution:		
Bonn University, Germany	Physics, Mathematics	B.Sc. (Vordiplom) 1991
Graduate Institution:		
S.U.N.Y. Stony Brook, U.S.A.	Mathematical Physics	Ph.D. 1997
Postdoctoral Institutions:		
Amsterdam University, Holland	Mathematical Physics	Postdoctoral Fellow 1997-1999
M.I.T., U.S.A	Algebra	C.L.E. Moore Instructor 1999-2001

b. Appointments

Full Professor	since July 2006	Mathematics Department at UC Davis
Associate Professor	July 2004-June 2006	Mathematics Department at UC Davis
Assistant Professor	July 2000-June 2004	Mathematics Department at UC Davis
Research Scientist	July-August 2001	Max-Planck-Institut für Mathematik in Bonn
Humboldt Research Fellow	June-December 2002	University of Wuppertal
Research Scientist	June-September 2003	Max-Planck-Institut für Mathematik in Bonn

c. Selected Publications

(i) Five recent papers closely related to the proposed research:

1. T. Lam, A. Schilling, M. Shimozono, K-theory Schubert calculus of the affine Grassmannian, Compositio Mathematica **146** Issue 4 (2010) 811–852
2. G. Fourier, M. Okado, A. Schilling, Kirillov-Reshetikhin crystals for nonexceptional types, Advances in Mathematics **222** Issue 3 (2009) 1080–1116
3. J. Bandlow, A. Schilling, M. Zabrocki, The Murnaghan-Nakayama rule for k-Schur functions, Journal of Combinatorial Theory, Series A, **118**(5) (2011) 1588–1607
4. A. Schilling Combinatorial structure of Kirillov-Reshetikhin crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, J. Algebra **319** (2008) 2938-2962.
5. J. Bandlow, A. Schilling, N. M. Thiéry, On the uniqueness of promotion operators on tensor products of type A crystals, J. Algebraic Combinatorics **31** (2010) 217–251

(ii) Five other significant papers:

1. M. Okado, A. Schilling, Existence of Kirillov-Reshetikhin crystals for nonexceptional types, Representation Theory **12** (2008) 186–207.
2. G. Fourier, A. Schilling, M. Shimozono, Demazure structure inside Kirillov-Reshetikhin crystals, J. Algebra **309** (2007) 386–404
3. A. Schilling, Crystal structure on rigged configurations, International Mathematics Research Notices, Volume 2006, Article ID 97376, Pages 1-27.

4. F. Hivert, A. Schilling, N. M. Thiéry, Hecke group algebras as quotients of affine Hecke algebras at level 0, Journal of Combinatorial Theory, Series A **116** (2009) 844–863.

5. M. Okado, A. Schilling, M. Shimozono, Virtual crystals and Kleber's algorithm, Commun. Math. Phys. 238 (2003) 187–209.

d. Synergistic Activities

- Advisor of graduate students and postdocs: advisor of five PhD students and three senior theses/projects with undergraduate students, currently advising at least two PhD students;
  Mentored the following postdocs: Jason Bandlow, Andrew Berget, Ghislain Fourier (at MSRI), Brant Jones, Rahbar Virk, Sankaran Viswanath, Alex Woo.

- Undergraduate Program Committee Chair: responsible for the restructuring of the entire undergraduate mathematics class offering at UC Davis. The new program went into effect Fall 2006. Joint on-line textbook for the linear algebra class MAT 67 with Bruno Nachtergaele and Isaiah Lankham.

- Coorganizer of the semester long program on "Combinatorial representation theory" held at MSRI January through May 2008, in particular the Introductory Workshop for this program in January 2008, an AMS special session in Claremont in May 2008, a Square Workshop at AIM in May 2009, summer school and workshop on "Affine Schubert calculus" at the Fields Institute in July 2010.

- Contributed code on crystal graphs to the open-source Mathematical Software system `Sage` and is one of the main developers for `Sage-Combinat`.

- During the last year Schilling presented many seminar and colloquium talks, in particular with respect to SAGE, for example at the international conference FPSAC'10 in San Francisco, August 2010; Sage Days at the Fields Institute in Toronto, May 2010; workshop on "Whittaker Functions, Crystal Bases, and Quantum Groups at Banff", June 2010; Sage Days 26 in Seattle, December 2010.

e. Collaborators and Other Affiliations

(i) Collaborators (within the last five years):

Jason Bandlow (University of Pennsylvania), Lipika Deka (California State University Monterey Bay), Tom Denton (UC Davis), Ghislain Fourier (Universität Köln, Germany), Florent Hivert (Université Rouen, France), Brant Jones (James Madison University), Thomas Lam (University of Michigan), Cristian Lenart (SUNY Albany), Jean-Christophe Novelli (Marne-la-Vallé, France) Masato Okado (Osaka University, Japan), Mark Shimozono (Virginia Tech), Philip Sternberg (UC Davis), Nicolas M. Thiéry (Université Paris-Sud, Orsay, France), Peter Tingley (MIT), Qiang Wang (UC Davis), Mike Zabrocki (York University, Canada)

(ii) Graduate and Postdoctoral Advisors:

Barry M. McCoy, State University of New York at Stony Brook, U.S.A.
Bernard Nienhuis, University of Amsterdam, The Netherlands

(iii) Thesis Advisor and Postgraduate-Scholar Sponsor:

Jason Bandlow (University of Pennsylvania), Andrew Berget (UC Davis), Lipika Deka (California State University, Monterey Bay), Tom Denton (UC Davis), Ghislain Fourier (Universität Köln), Brant Jones (James Madison University), Steven Pon (University of Connecticut), Philip Sternberg (IBM), Rahbar Virk (UC Davis), Sankaran Viswanath (Indian Institute of Science), Qiang Wang (UC Davis)

Biographical Sketch: Gregg Musiker

(a) Professional Preparation

Harvard University Mathematics, A.B. *Magna Cum Laude*, 1998-2002

University of California, San Diego Mathematics, Ph.D., 2002-2007

Massachusetts Institute of Technology Applied Mathematics, NSF Postdoctoral Fellow and Instructor in Applied Mathematics, 2007-2010

Mathematical Sciences Research Institute MSRI Postdoctoral Fellow, program: *Tropical Geometry*, Fall 2009

(b) Appointments

2010–present Assistant Professor, University of Minnesota
2009 (Fall Semester) MSRI Postdoctoral Fellow, program: *Tropical Geometry*
2007–2010 NSF Postdoctoral Fellow and Instructor in Applied Mathematics, MIT

(c)(i) Publications most closely related to the proposed project

1. G. Musiker and C. Stump, A compendium on the cluster algebra and quiver package in SAGE, submitted in Mar. 2011; `http://arxiv.org/pdf/1102.4844.pdf`
2. G. Musiker, R. Schiffler, and L. Williams, Positivity for cluster algebras from surfaces, *Adv. Math.*, **227** (2011), Issue 6, 2241–2308.
3. G. Musiker and R. Schiffler, Cluster expansion formulas and perfect matchings *J. Algebraic Combin.* **32** (2010), Issue 2, 187-209.
4. G. Musiker, A graph theoretic expansion formula for cluster algebras of classical type *Ann. of Combin.*, **15** (2011), 147–184.
5. G. Musiker and J. Propp, Combinatorial interpretations for rank-two cluster algebras of affine type *Electron. J. Combin.* **14** (2007), no. 1, Research Paper 15, 1-23.

(c)(ii) Other significant publications

1. G. Musiker and V. Reiner, The cyclotomic polynomial topologically, submitted in Jan. 2011; `http://arxiv.org/pdf/1012.1844.pdf`
2. C. Haase, G. Musiker, and J. Yu, Linear systems on tropical curves, *Math. Zeitschrift*, online Feb. 2011; `http://dx.doi.org/10.1007/s00209-011-0844-4`

3. A. Garsia, G. Musiker, N. Wallach, and G. Xin, Invariants, Kronecker products and combinatorics of some remarkable Diophantine systems *Adv. in Appl. Math.* **42** (2009), no. 3, 392-421.

4. G. Musiker, The critical groups of a family of graphs and elliptic curves over finite fields *J. Algebraic Combin.* **30** (2009), Issue 2, 255-276.

5. J. Bandlow and G. Musiker, A new characterization for the m-quasi-invariants of S_n and explicit basis for two row hook shapes *J. Combin. Theory Ser. A* **115** (2008), no. 8, 1333-1357.

(d) Synergistic Activities

- *Organizer* and Lecturer for MSRI Graduate Summer School on "Cluster Algebras and Cluster Combinatorics" (with Lauren Williams)
- *Developer* of Cluster Algebra software for Sage (with Christian Stump)
- *Supervisor* of Undergraduate Research: Eleven REU students at University of Minnesota (with Pavlo Pylyavskyy and Vic Reiner) in Summer 2011; Andrei Frimu as an MIT UROP in Summer 2009; Anthony Kim as an MIT UROP (with Josephine Yu) in Spring 2008
- *Author* of OCW (Open Course Ware) Lecture notes: Publishing lecture notes from Spring 2009 Course in Algebraic Combinatorics through MIT's OCW repository, which allows open access.
- Journals Refereed: *Acta Mathematica Sinica*, *Advances in Mathematics*, *The Australasian Journal of Combinatorics*, *Discrete Mathematics and Theoretical Computer Science*, *Electronic Journal of Combinatorics*, *Journal of Algebraic Combinatorics*, *Journal of Combinatorial Theory, Series A*, *Linear and Multilinear Algebra*, *Qaestiones Mathematicae*, *Psi-Mu-Epsilon Journal*; Grants Refereed: *NSA*

(e) Collaborators and other affiliates

(i) *Collaborators:* Jason Bandlow (U. Penn.), Adriano Garsia (UCSD), Christian Haase (FU-Berlin), Jim Propp (U. Mass. Lowell), Vic Reiner (U. Minnesota), Ralf Schiffler (U. Conn.), Christian Stump (UQAM), Nolan Wallach (UCSD), Lauren Williams (UC Berkeley), Guoce Xin (Nankai Univ.), Josephine Yu (MSRI/Georgia Tech.)

(ii) *Advisors:* Undergraduate Thesis Advisor and Postdoctoral Advisor: Richard Stanley (MIT), Ph.D. Advisor: Adriano Garsia (UCSD)

(iii) *Doctoral students advised:* Alex Garver, Emily Gunawan

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1 Biographical Sketch: Daniel Bump

1.0.1 Professional Preparation

BA (Mathematics):	Reed College, 1974
PhD (Mathematics):	University of Chicago, 1982

1.0.2 Academic Appointments

1995–	Professor	Stanford University
1990–1995	Associate Professor	Stanford University
1986–1990	Assistant Professor	Stanford University
1985–1986	Member	The Institute for Advanced Study, Princeton
1983–1985	Lecturer	The University of Texas at Austin

1.0.3 Publications most closely related to this project

Recent papers are available at `http://math.stanford.edu/~bump/`

1. B. Brubaker, D. Bump, S. Friedberg, Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory, Annals of Mathematics Studies 175, 2011.

2. B. Brubaker, D. Bump, S. Friedberg, Weyl group multiple Dirichlet series, Eisenstein series and crystal bases, Ann. of Math. 2011.

3. D. Bump and M. Nakasuji, The Casselman Basis of Iwahori Fixed Vectors and the Bruhat order, Canadian Journal of Mathematics, to appear. `http://arxiv.org/abs/1002.2996`

4. B. Brubaker, D. Bump, S. Friedberg, Schur Polynomials and the Yang-Baxter Equation. Comm. Math. Phys., to appear. `http://arxiv.org/abs/0912.0911`

5. B. Brubaker, D. Bump, G. Chinta, S. Friedberg, P. Gunnells. *Metaplectic Ice*. Submitted. `http://arxiv.org/abs/1009.1741`

1.0.4 Other Significant Publications

1. D Bump and S. Friedberg, *Weyl group multiple Dirichlet series II: the stable case*, Invent. Math. 165:325–355 (2006).

2. D. Bump and P. Diaconis, *Toeplitz Minors*, J. Combin. Theory Ser. A **97** (2002), 252–271.

3. D. Bump and A. Gamburd, *On the Averages of Characteristic Polynomials from Classical Groups,* Comm. Math. Phys. 265:227–274 (2006).

4. B. Brubaker, D. Bump, S. Friedberg, Gauss sum combinatorics and metaplectic Eisenstein series, in AMS Contemp. Math. **488**, 2009.

5. B. Brubaker, D. Bump, S. Friedberg and J. Hoffstein, *Weyl group multiple Dirichlet series III: twisted unstable* A_r, Ann. of Math., (2006).

2 Synergistic Activities: Daniel Bump

1. The investigator is the author of texts *Automorphic Forms and Representations* (Cambridge) and *Lie Groups*, Springer.

2. Has mentored thirteen graduate students, directed eight senior theses at Stanford and co-advised another at Reed College.

3. In 2009, gave lectures on Schur polynomials and the Yang-Baxter equation in Göttingen and in Weihai, attended by many graduate students.

4. Helped organize workshops in related areas in 2005, 2006, 2009 and 2010, and helped give a mini-course on crystal bases and multiple Dirichlet series in Edinburg, 2008.

5. Contributor to SAGE Mathematical Software, methods for Lie groups, crystal bases and combinatorics.

3 Collaborators and Affiliations: Daniel Bump

Recent collaborators: Jennifer Beineke (Western New England), Ben Brubaker (MIT), Gautam Chinta (CUNY), YoungJu Choi (Pohang), Persi Diaconis (Stanford), Solomon Friedberg (Boston College), Paul Gunnells (UMass Amherst) Alex Gamburd (Santa Cruz), David Ginzburg (Tel Aviv), Jeffrey Hoffstein (Brown), Anthony Licata (Stanford), Maki Nakasuji (Stanford), Peter J. McNamara (Stanford).

Graduate Advisors: Walter Baily (Chicago), advisor; Joe Buhler (Reed College and IDA), unofficial graduate mentor.

Recent PhD Advisees: Ryan Vinroot (William and Mary), Edray Goins (Purdue), Yiannis Sakellaridis (Rutgers Newark), Paul Dehaye (ETH), David Lecomte, Dmitriy Ivanov. Total PhD advisees: 13

Biographical Sketch: William Stein

Professional Preparation

Harvard University	NSF Postdoc, 2000–2004
University of California at Berkeley	Mathematics, Ph.D. 2000
Northern Arizona University	Mathematics, B.S. 1994

Appointments

- Professor of Mathematics (with tenure), University of Washington, September 2010–present.
- Associate Professor of Mathematics (with tenure), University of Washington, September 2006–2010.
- Associate Professor of Mathematics (with tenure), UC San Diego, July 2005–June 2006.
- Benjamin Peirce Assistant Professor of Mathematics, Harvard University, July 2001–May 2005.
- NSF Postdoctoral Research Fellowship under Barry Mazur at Harvard University, August 2000–May 2004.
- Clay Mathematics Institute Liftoff Fellow, Summer 2000.

Most Relevant Publications

- *The Sage Project: Unifying Free Mathematical Software to Create a Viable Alternative to Magma, Maple, Mathematica and MATLAB* (16 pages), 2010, in the Proceedings of the International Congress of Mathematical Software, Kobe, Japan.
- *Toward a Generalization of the Gross-Zagier Conjecture* (17 pages), 2010, Int. Math. Res. Notices.
- *Fast Computation of Hermite Normal Forms of Random Integer Matrices* (16 pages), with Clement Pernet, Volume 130, Issue 7, July 2010, Pages 1675-1683, Journal of Number Theory.
- *Average Ranks of Elliptic Curves: Tension Between Data and Conjecture*, with B. Bektemirov, B. Mazur, and M. Watkins, Bulletins of the AMS **44** (2007), no. 2, 233–254.
- *Modular forms, a computational approach* (xvi+268 pp.) Graduate Studies in Mathematics (AMS) 79 2007, with an appendix by Paul Gunnells.

Biographical Sketch: William Stein

Other Publications

- *Computation of p-Adic Heights and Log Convergence*, with B. Mazur and J. Tate (36 pages), Documenta Mathematica, 2006, Extra Vol., 577–614.
- *The Manin Constant*, with A. Agashe and K. Ribet, Pure Appl. Math., (2006), no. 2., 617–636.
- *Verification of the Birch and Swinnerton-Dyer Conjecture for Specific Elliptic Curves*, with G. Grigorov, A. Jorza, S. Patrikis, and C. Patrascu (26 pages), 2005, to appear in Mathematics of Computation.
- *Shafarevich-Tate Groups of Nonsquare Order*, Progress in Math., **224** (2004), 277–289, Birkhauser.
- $J_1(p)$ *has connected fibers*, with B. Conrad and B. Edixhoven, Documenta Math., **8** (2003), 331–408.

Synergistic Activities

- **Research Tools:** Principal author of Sage, which is a major new piece of software (over 100,000 lines of code by the PI). Author of the modular forms, modular symbols, and modular abelian varieties parts of the Magma computer algebra system (425 pages (26000 lines) of code plus documentation).
- **Databases:** Created and maintain the Modular Forms Database. This contains continually expanding data about elliptic curves and modular forms: `http://www.wstein.org/Tables/`.
- **Outreach:** SIMUW 2006, 2007, 2008; Canada/USA MathCamp mentor (2002); Several Math Circles talks in Boston.

Collaborators and Other Affiliations

- **Coauthors:** A. Agashe (Florida State U.), K. Buzzard (Imperial College, London), R. Coleman (UC Berkeley), B. Conrad (Univ. of Michigan), N. Dummigan (Sheffield, UK), S. Edixhoven (Leiden, Netherlands), F. Leprévost (Univ. Joseph Fourier, Technische Univ. Berlin), E. V. Flynn (Liverpool, UK), D. Kohel (Univ. of Sydney), B. Mazur (Harvard), L. Merel (Paris 6), K. Ribet (UC Berkeley), E. F. Schaefer (Santa Clara Univ.), M. Stoll (Inter. Univ. Bremen, Germany), J. Tate, H. A. Verrill (Louisiana State), M. Watkins (Bristol.), J. L. Wetherell (CCR, San Diego), C. Wuthrich (Nottingham)
- **Graduate and Postdoctoral Advisors:**
  - **Ph.D. advisor:** Hendrik Lenstra, University of Leiden, Netherlands.
  - **NSF Postdoctoral advisor:** Barry Mazur, Harvard University.
- **Thesis Students:** 3 Ph.D. students at UW: Robert Bradshaw's 2010 Ph.D. on *Provable Computation of Motivic L-functions*; Robert Miller's 2010 Ph.D. on *Computational Verification of the Birch and Swinnerton-Dyer Conjecture*; currently advising Alyson Dienes's Ph.D. thesis. Advised eight undergraduate senior theses at Harvard and two at UW.