Homework Set 1: Exercises on Complex Numbers

Directions: You are assigned the **Calculational Problems** 1(a, b, c), 2(b), 3(a, b), 4(b, c), 5(a, b), and the **Proof-Writing Problems** 8 and 11.
Please submit your solutions to the Calculational and Proof-Writing Problems **separately** at the beginning of lecture on Friday January 12, 2007. The two sets will be graded by different persons.

1. Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:

   (a) $(2 + 3i) + (4 + i)$

   (b) $(2 + 3i)^2(4 + i)$

   (c) $\dfrac{2 + 3i}{4 + i}$

   (d) $\dfrac{1}{i} + \dfrac{3}{1 + i}$

   (e) $(-i)^{-1}$

2. Compute the real and imaginary parts of the following expressions, where z is the complex number $x + yi$ and $x, y \in \mathbb{R}$:

   (a) $\dfrac{1}{z^2}$

   (b) $\dfrac{1}{3z + 2}$

   (c) $\dfrac{z + 1}{2z - 5}$

   (d) z^3

3. Solve the following equations for z a complex number:

   (a) $z^5 - 2 = 0$

   (b) $z^4 + i = 0$

   (c) $z^6 + 8 = 0$

(d) $z^3 - 4i = 0$

4. Calculate the

   (a) complex conjugate of the fraction $(3+8i)^4/(1+i)^{10}$.
   (b) complex conjugate of the fraction $(8-2i)^{10}/(4+6i)^5$.
   (c) complex modulus of the fraction $i(2+3i)(5-2i)/(-2-i)$.
   (d) complex modulus of the fraction $(2-3i)^2/(8+6i)^2$.

5. Compute the real and imaginary parts:

   (a) e^{2+i}
   (b) $\sin(1+i)$
   (c) e^{3-i}
   (d) $\cos(2+3i)$

6. Compute the real and imaginary part of e^{e^z} for $z \in \mathbb{C}$.

7. Let $a \in \mathbb{R}$ and $z, w \in \mathbb{C}$. Prove that

   (a) $\mathrm{Re}(az) = a\mathrm{Re}(z)$ and $\mathrm{Im}(az) = a\mathrm{Im}(z)$.
   (b) $\mathrm{Re}(z+w) = \mathrm{Re}(z) + \mathrm{Re}(w)$ and $\mathrm{Im}(z+w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.

8. Let $z \in \mathbb{C}$. Prove that $\mathrm{Im}(z) = 0$ if and only if $\mathrm{Re}(z) = z$.

9. Let p be a polynomial with real coefficients and $z \in \mathbb{C}$.
   Show that $p(z) = 0$ if and only $p(\overline{z}) = 0$.

10. Let $z, w \in \mathbb{C}$. Prove the *parallelogram law* $|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$.

11. Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$ such that either $|z| = 1$ or $|w| = 1$. Prove that

$$\left| \frac{z - w}{1 - \overline{z}w} \right| = 1.$$

12. For an angle $\theta \in [0, 2\pi)$, find the linear map $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which describes the rotation by the angle θ in the counterclockwise direction.
Hint: For a given angle θ find a, b, c, d such that $f_\theta(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$.