Homework Set 2: Exercises on Linear Equations and Vector Spaces

Directions: Submit your solutions to Problems 1, 2 and 4. Separately, please also submit the Proof-Writing-Problems 3 and 5. This homework is due on Friday January 19, 2007 at the beginning of lecture.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1. Solve the following systems of linear equations and characterize their solution set (unique solution, no solution, ...). Also write each system of linear equations as an equation for a single function $f : \mathbb{R}^n \to \mathbb{R}^m$ for appropriate m, n.

   (a) System of 3 equations in the unknowns x, y, z, w

   $$\begin{aligned} x + 2y - 2z + 3w &= 2 \\ 2x + 4y - 3z + 4w &= 5 \\ 5x + 10y - 8z + 11w &= 12. \end{aligned}$$

   (b) System of 4 equations in the unknowns x, y, z

   $$\begin{aligned} x + 2y - 3z &= 4 \\ x + 3y + z &= 11 \\ 2x + 5y - 4z &= 13 \\ 2x + 6y + 2z &= 22. \end{aligned}$$

   (c) System of 3 equations in the unknowns x, y, z

   $$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2. \end{aligned}$$

2. Show that the space $V = \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 + 2x_3 = 0\}$ forms a vector space.

3. Let V be a vector space over $\mathbb{F}$. Then, given $a \in \mathbb{F}$ and $v \in V$ such that $av = 0$, prove that either $a = 0$ or $v = 0$.

4. Give an example of a nonempty subset $U \subset \mathbb{R}^2$ such that U is closed under scalar multiplication but is not a subspace of $\mathbb{R}^2$.

5. Let V be a vector space over $\mathbb{F}$, and suppose that W_1 and W_2 are subspaces of V. Prove that their intersection $W_1 \cap W_2$ is also a subspace of V.

6. Prove or give a counterexample to the following claim:

   Claim. *Let V be a vector space over $\mathbb{F}$, and suppose that W_1, W_2, and W_3 are subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.*

7. Let $\mathbb{F}[z]$ denote the vector space of all polynomials having coefficient over $\mathbb{F}$, and define U to be the subspace of $\mathbb{F}[z]$ given by

   $$U = \{az^2 + bz^5 \mid a, b \in \mathbb{F}\}.$$

   Find a subspace W of $\mathbb{F}[z]$ such that $\mathbb{F}[z] = U \oplus W$.

8. Prove or give a counterexample to the following claim:

   Claim. *Let V be a vector space over $\mathbb{F}$, and suppose that W_1, W_2, and W_3 are subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.*