Homework Set 8: Exercises on Inner Product Spaces

Directions: Please work on all of the following exercises and then submit your solutions to the Calculational Problems 1 and 8, and the Proof-Writing Problems 2 and 11 at the **beginning** of lecture on March 2, 2007.

As usual, we are using $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. We also use $\langle \cdot , \cdot \rangle$ to denote an arbitrary inner product and $\| \cdot \|$ to denote its associated norm.

1. Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, and define

$$f_1 = e_1 + e_2 + e_3$$
$$f_2 = e_2 + e_3$$
$$f_3 = e_3.$$

   (a) Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3).

   (b) What would you obtain if you applied the Gram-Schmidt process to the basis (f_3, f_2, f_1)?

2. Let V be a finite-dimensional inner product space over $\mathbb{F}$. Given any vectors $u, v \in V$, prove that the following two statements are equivalent:

   (a) $\langle u, v \rangle = 0$

   (b) $\|u\| \leq \|u + \alpha v\|$ for every $\alpha \in \mathbb{F}$.

3. Let $n \in \mathbb{Z}_+$ be a positive integer, and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ be any collection of $2n$ real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \left(\sum_{k=1}^{n} k a_k^2 \right) \left(\sum_{k=1}^{n} \frac{b_k^2}{k} \right)$$

4. Prove or disprove the following claim:

   Claim. *There is an inner product $\langle \cdot , \cdot \rangle$ on $\mathbb{R}^2$ whose associated norm $\| \cdot \|$ is given by the formula*

$$\|(x_1, x_2)\| = |x_1| + |x_2|$$

   for every vector $(x_1, x_2) \in \mathbb{R}^2$, where $| \cdot |$ denotes the absolute value function on $\mathbb{R}$.

5. Let V be a finite-dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that
$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

6. Let V be a finite-dimensional inner product space over $\mathbb{C}$. Given $u, v \in V$, prove that
$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4} + \frac{\|u+iv\|^2 - \|u-iv\|^2}{4}i.$$

7. Let $\mathcal{C}[-\pi, \pi] = \{f : [-\pi, \pi] \to \mathbb{R} \mid f \text{ is continuous}\}$ denote the inner product space of continuous real-valued functions defined on the interval $[-\pi, \pi] \subset \mathbb{R}$, with inner product given by
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx, \text{ for every } f, g \in \mathcal{C}[-\pi, \pi].$$
Then, given any positive integer $n \in \mathbb{Z}_+$, prove that the set of vectors
$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \ldots, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \ldots, \frac{\cos(nx)}{\sqrt{\pi}}\right\}$$
is orthonormal.

8. Let $\mathbb{R}_2[x]$ denote the inner product space of polynomials over $\mathbb{R}$ having degree at most two, with inner product given by
$$\langle f, g\rangle = \int_{0}^{1} f(x)g(x)dx, \text{ for every } f, g \in \mathbb{R}_2[x].$$
Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathbb{R}_2[x]$ in order to produce an orthonormal basis for $\mathbb{R}_2[x]$.

9. Let V be a finite-dimensional inner product space over $\mathbb{F}$, and let U be a subspace of V. Prove that the orthogonal complement $U^\perp$ of U with respect to the inner product $\langle\cdot\,,\cdot\rangle$ on V satisfies
$$\dim(U^\perp) = \dim(V) - \dim(U).$$

10. Let V be a finite-dimensional inner product space over $\mathbb{F}$, and let U be a subspace of V. Prove that $U = V$ if and only if the orthogonal complement $U^\perp$ of U with respect to the inner product $\langle\cdot\,,\cdot\rangle$ on V satisfies $U^\perp = \{0\}$.

11. Let V be a finite-dimensional inner product space over $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ is a linear operator on V having the following two properties:

    (a) Given any vector $v \in V$, $P(P(v)) = P(v)$. I.e., $P^2 = P$.

    (b) Given any vector $u \in \text{null}(P)$ and any vector $v \in \text{range}(P)$, $\langle u, v \rangle = 0$.

    Prove that P is an orthogonal projection.