Vector Spaces

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1 Definition of vector spaces

As we have seen in the introduction, a vector space is a set V with two operations: addition of vectors and scalar multiplication. These operations satisfy certain properties, which we are about to discuss in more detail. The scalars are taken from a field $\mathbb{F}$, where for the remainder of these notes $\mathbb{F}$ stands either for the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The real and complex numbers are examples of fields. The abstract definition of a field and further examples are studied in algebra courses, such as the MAT 150 series.

Vector addition can be thought of as a map $+ : V \times V \to V$, mapping two vectors $u, v \in V$ to their sum $u + v \in V$. Scalar multiplication can be described as a map $\mathbb{F} \times V \to V$, which assigns to a scalar $a \in \mathbb{F}$ and a vector $v \in V$ a new vector av.

Definition 1. A **vector space** over $\mathbb{F}$ is a set V together with the operations of addition $V \times V \to V$ and scalar multiplication $\mathbb{F} \times V \to V$ satisfying the following properties:

1. **Commutativity:** $u + v = v + u$ for all $u, v \in V$;
2. **Associativity:** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and $a, b \in \mathbb{F}$;
3. **Additive identity:** There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$;
4. **Additive inverse:** For every $v \in V$, there exists an element $w \in V$ such that $v+w = 0$;
5. **Multiplicative identity:** $1v = v$ for all $v \in V$;
6. **Distributivity:** $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $u, v \in V$ and $a, b \in \mathbb{F}$.

Usually, a vector space over $\mathbb{R}$ is called a **real vector space** and a vector space over $\mathbb{C}$ is called a **complex vector space**. The elemens $v \in V$ of a vector space are called **vectors**.

Vector spaces are very fundamental objects in mathematics. Definition 1 is an abstract definition, but there are many examples of vector spaces. You will see many examples of vector spaces throughout your mathematical life. Here are just a few:

Example 1. Consider the set $\mathbb{F}^n$ of all n-tuples with elements in $\mathbb{F}$. This is a vector space. Addition and scalar multiplication are defined componentwise. That is, for $u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in \mathbb{F}^n$ and $a \in \mathbb{F}$, we define

$$\begin{aligned} u + v &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \\ au &= (au_1, au_2, \ldots, au_n). \end{aligned}$$

It is easy to check that all properties of Definition 1 are satisfied. In particular the additive identity $0 = (0, 0, \ldots, 0)$ and the additive inverse of u is $-u = (-u_1, -u_2, \ldots, -u_n)$.

Special cases of Example 1 are $\mathbb{R}^n$, in particular $\mathbb{R}^2$ and $\mathbb{R}^3$. We have already seen in the introduction that there is a geometric interpretation for elements in $\mathbb{R}^2$ and $\mathbb{R}^3$ as points in the plane and 3-space, respectively.

Example 2. Let $\mathbb{F}^\infty$ be the set

$$\mathbb{F}^\infty = \{(u_1, u_2, \ldots) \mid u_j \in \mathbb{F} \text{ for } j = 1, 2, \ldots\}.$$

Addition and scalar multiplication are defined as expected

$$\begin{aligned} (u_1, u_2, \ldots) + (v_1, v_2, \ldots) &= (u_1 + v_1, u_2 + v_2, \ldots), \\ a(u_1, u_2, \ldots) &= (au_1, au_2, \ldots). \end{aligned}$$

You should verify that with these operations $\mathbb{F}^\infty$ becomes a vector space.

Example 3. Verify that $V = \{0\}$ is a vector space!

Example 4. Let $\mathcal{P}(\mathbb{F})$ be the set of all polynomials $p : \mathbb{F} \to \mathbb{F}$ with coefficients in $\mathbb{F}$. More precisely, $p(z)$ is a polynomial if there exist $a_0, a_1, \ldots, a_n \in \mathbb{F}$ such that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0. \tag{1}$$

Addition and scalar multiplication are defined as

$$\begin{aligned} (p + q)(z) &= p(z) + q(z), \\ (ap)(z) &= ap(z), \end{aligned}$$

where $p, q \in \mathcal{P}(\mathbb{F})$ and $a \in \mathbb{F}$. For example, if $p(z) = 5z + 1$ and $q(z) = 2z^2 + z + 1$, then $(p + q)(z) = 2z^2 + 6z + 2$ and $(2p)(z) = 10z + 2$. Again, it can be easily verified that

$\mathcal{P}(\mathbb{F})$ forms a vector space over $\mathbb{F}$. The additive identity in this case is the zero polynomial, for which all coefficients are equal to zero. The additive inverse of $p(z)$ in (1) is $-p(z) = -a_n z^n - a_{n-1} z^{n-1} - \cdots - a_1 z - a_0$.

2 Elementary properties of vector spaces

We are going to prove several important, yet simple properties of vector spaces. From now on V will denote a vector space over $\mathbb{F}$.

Proposition 1. *Every vector space has a unique additive identity.*

Proof. Suppose there are two additive identities 0 and $0'$. Then

$$0' = 0 + 0' = 0,$$

where the first equality holds since 0 is an identity and the second equality holds since $0'$ is an identity. Hence $0 = 0'$ proving that the additive identity is unique. □

Proposition 2. *Every $v \in V$ has a unique additive inverse.*

Proof. Suppose w and w' are additive inverses of v, so that $v + w = 0$ and $v + w' = 0$. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Hence $w = w'$ as desired. □

Since the additive inverse of v is unique as just shown, it will from now on be denoted by $-v$. We define $w - v$ to mean $w + (-v)$. We will in fact show in Proposition 5 that $-v = -1v$.

Proposition 3. $0v = 0$ *for all* $v \in V$.

Note that the 0 on the left hand side in Proposition 3 is a scalar, whereas the 0 on the right hand side is a vector.

Proof. For $v \in V$ we have

$$0v = (0 + 0)v = 0v + 0v,$$

using distributivity. Adding the additive inverse of $0v$ to both sides we obtain

$$0 = 0v - 0v = (0v + 0v) - 0v = 0v.$$

□

Proposition 4. $a0 = 0$ *for every* $a \in \mathbb{F}$.

Proof. Similarly to the proof of Proposition 3, we have for $a \in \mathbb{F}$

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides we obtain $0 = a0$ as desired. □

Proposition 5. *$(-1)v = -v$ for every $v \in V$.*

Proof. For $v \in V$ we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0,$$

which shows that $(-1)v$ is the additive inverse $-v$ of v. □

3 Subspaces

Definition 2. A subset $U \subset V$ of a vector space V over $\mathbb{F}$ is a **subspace** of V if U itself is a vector space over $\mathbb{F}$.

To check that a subset $U \subset V$ is a subspace, it suffices to check only a couple of the conditions of a vector space.

Lemma 6. *Let $U \subset V$ be a subset of a vector space V over $\mathbb{F}$. Then U is a subspace of V if and only if*

1. ***additive identity:*** $0 \in U$;
2. ***closure under addition:*** $u, v \in U$ *implies* $u + v \in U$;
3. ***closure under scalar multiplication:*** $a \in \mathbb{F}$, $u \in U$ *implies that* $au \in U$.

Proof. 1 implies that the additive identity exists. 2 implies that vector addition is well-defined and 3 ensures that scalar multiplication is well-defined. All other conditions for a vector space are inherited from V since addition and scalar multiplication for elements in U are the same viewed as elements in U or V. □

Example 5. In every vector space V, the subsets $\{0\}$ and V are trivial subspaces.

Example 6. $\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

Example 7. $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 = 0\}$ is a subspace of $\mathbb{F}^3$. To see this we need to check the three conditions of Lemma 6. The zero vector $(0, 0, 0) \in \mathbb{F}^3$ is in U since it satisfies the condition $x_1 + 2x_2 = 0$. To show that U is closed under addition, take two vectors $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$. Then by the definition of U we have $v_1 + 2v_2 = 0$

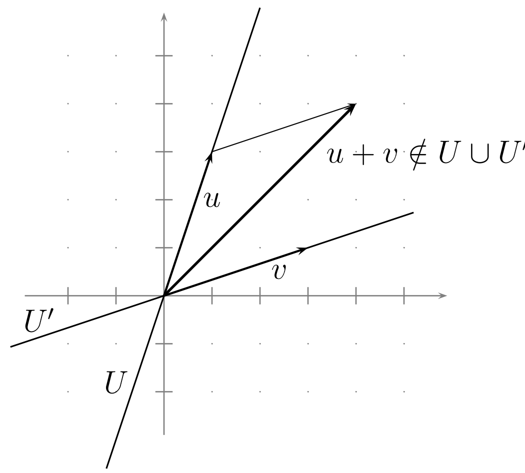

Figure 1: The union $U \cup U'$ of two subspaces is not necessarily a subspace

and $u_1 + 2u_2 = 0$. Adding these two equations it is not hard to see that then the vector $v + u = (v_1 + u_1, v_2 + u_2, v_3 + u_3)$ satisfies $(v_1 + u_1) + 2(v_2 + u_2) = 0$. Hence $v + u \in U$. Similarly, to show closure under scalar multiplication, take $u = (u_1, u_2, u_3) \in U$ and $a \in \mathbb{F}$. Then $au = (au_1, au_2, au_3)$ satisfies the equation $au_1 + 2au_2 = a(u_1 + 2u_2) = 0$, so that $au \in U$.

Example 8. $\{p \in \mathcal{P}(\mathbb{F}) \mid p(3) = 0\}$ is a subspace of $\mathcal{P}(\mathbb{F})$.

Example 9. The subspaces of $\mathbb{R}^2$ are $\{0\}$, all lines through the origin, and $\mathbb{R}^2$. The subspaces of $\mathbb{R}^3$ are $\{0\}$, all lines through the origin, all planes through the origin, and $\mathbb{R}^3$. In fact, these exhaust all subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. To prove this we will need further tools such as the notion of bases and dimensions to be discussed soon. In particular this shows that lines or planes that do not pass through the origin are not subspaces (this is not so hard to show!).

For all examples above, check that the conditions of Lemma 6 are satisfied.

Note that if U and U' are subspaces of V, then their intersection $U \cap U'$ is also a subspace (see Homework 2 and Figure 2). However, the union of two subspaces is not necessarily a subspace. Think for example of the union of two lines in $\mathbb{R}^2$, see Figure 1.

4 Sums and direct sums

Throughout this section V is a vector space over $\mathbb{F}$ and $U_1, U_2 \subset V$ denote subspaces.

Definition 3. Let $U_1, U_2 \subset V$ be subspaces of V. Define the **sum** of U_1 and U_2 as

$$U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}.$$

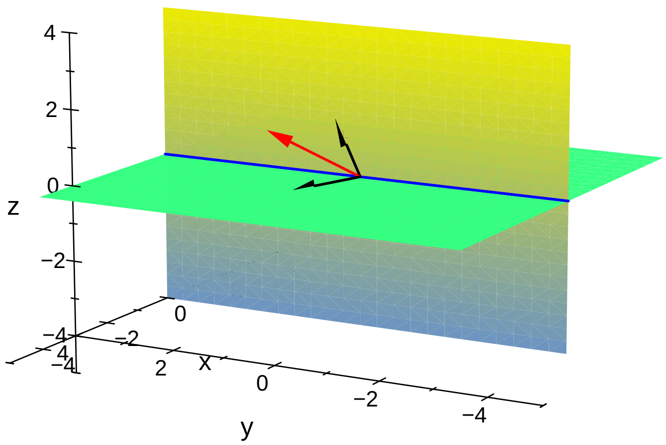

Figure 2: The intersection $U \cap U'$ of two subspaces is a subspace

Check as an exercise that $U_1 + U_2$ is a subspace of V. In fact, $U_1 + U_2$ is the smallest subspace of V that contains both U_1 and U_2.

Example 10. Let

$$U_1 = \{(x, 0, 0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\},$$
$$U_2 = \{(0, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}.$$

Then

$$U_1 + U_2 = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}. \tag{2}$$

If alternatively $U_2 = \{(y, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}$ then (2) still holds.

If $U = U_1 + U_2$, then for any $u \in U$ there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $u = u_1 + u_2$. If it so happens that u can be **uniquely** written as $u_1 + u_2$, then U is the direct sum of U_1 and U_2.

Definition 4. Suppose every $u \in U$ can be uniquely written as $u = u_1 + u_2$ for $u_1 \in U_1$ and $u_2 \in U_2$. Then

$$U = U_1 \oplus U_2$$

is the **direct sum** of U_1 and U_2.

Example 11. Let

$$U_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\},$$
$$U_2 = \{(0, 0, z) \in \mathbb{R}^3 \mid z \in \mathbb{R}\}.$$

Then $\mathbb{R}^3 = U_1 \oplus U_2$. However, if instead

$$U_2 = \{(0, w, z) \mid w, z \in \mathbb{R}\},$$

then $\mathbb{R}^3 = U_1 + U_2$, but it is not the direct sum of U_1 and U_2.

Example 12. Let

$$U_1 = \{p \in \mathcal{P}(\mathbb{F}) \mid p(z) = a_0 + a_2 z^2 + \cdots + a_{2m} z^{2m}\},$$
$$U_2 = \{p \in \mathcal{P}(\mathbb{F}) \mid p(z) = a_1 + a_3 z^3 + \cdots + a_{2m+1} z^{2m+1}\}.$$

Then $\mathcal{P}(\mathbb{F}) = U_1 \oplus U_2$.

Proposition 7. *Let $U_1, U_2 \subset V$ be subspaces. Then $V = U_1 \oplus U_2$ if and only if*

1. *$V = U_1 + U_2$;*
2. *If $0 = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$, then $u_1 = u_2 = 0$.*

Proof.

"$\Longrightarrow$" Suppose $V = U_1 \oplus U_2$. Then 1 holds by definition. Certainly $0 = 0 + 0$ and since by uniqueness this is the only way to write $0 \in V$ we have $u_1 = u_2 = 0$.

"$\Longleftarrow$" Suppose 1 and 2 hold. By 1 we have that for all $v \in V$ there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2$. Suppose $v = w_1 + w_2$ with $w_1 \in U_1$ and $w_2 \in U_2$. Subtract the two equations to obtain

$$0 = (u_1 - w_1) + (u_2 - w_2),$$

where $u_1 - w_1 \in U_1$ and $u_2 - w_2 \in U_2$. By 2 this implies $u_1 - w_1 = 0$ and $u_2 - w_2 = 0$, or equivalently $u_1 = w_1$ and $u_2 = w_2$ as desired. $\square$

Proposition 8. *Let $U_1, U_2 \subset V$ be subspaces. Then $V = U_1 \oplus U_2$ if and only if*

1. *$V = U_1 + U_2$;*
2. *$U_1 \cap U_2 = \{0\}$.*

Proof.

"$\Longrightarrow$" Suppose $V = U_1 \oplus U_2$. Then 1 holds by definition. If $u \in U_1 \cap U_2$, then $0 = u + (-u)$ with $u \in U_1$ and $-u \in U_2$ (why?). By Proposition 7 we have $u = 0$ and $-u = 0$, so that $U_1 \cap U_2 = \{0\}$.

"$\Longleftarrow$" Suppose 1 and 2 hold. To prove that $V = U_1 \oplus U_2$ holds, suppose that

$$0 = u_1 + u_2 \qquad \text{where } u_1 \in U_1 \text{ and } u_2 \in U_2. \tag{3}$$

By Proposition 7 it suffices to show that $u_1 = u_2 = 0$. Equation (3) implies that $u_1 = -u_2 \in U_2$. Hence $u_1 \in U_1 \cap U_2$ which in turn implies that $u_1 = 0$. Hence also $u_2 = 0$ as desired. $\square$

Everything in this section can be generalized to m subspaces $U_1, U_2, \ldots U_m$, except Proposition 8. To see this consider the following example:

Example 13. Let

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}, \\ U_2 &= \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}, \\ U_3 &= \{(0, y, y) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}. \end{aligned}$$

Then certainly $\mathbb{F}^3 = U_1 + U_2 + U_3$, but $\mathbb{F}^3 \neq U_1 \oplus U_2 \oplus U_3$ since for example $(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$. But $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$, so that the analogon of Proposition 8 does not hold.