Name: Student ID# :

MAT067 University of California, Davis Winter 2007

Sample Final Exam Problems

Problem 1. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be given by

$$\begin{aligned} v_1 &= (1,2,1) \\ v_2 &= (1,-2,1) \\ v_3 &= (1,2,-1) \end{aligned}$$

Apply the Gram-Schmidt process to the basis (v_1, v_2, v_3) of $\mathbb{R}^3$, and call the resulting o.n. basis (u_1, u_2, u_3).

Problem 2. Let $A \in \mathbb{C}^{3\times 3}$ be given by

$$A = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{pmatrix}$$

a) Calculate $\det A$.

b) Find $\det A^4$.

Problem 3. Let $P \subset \mathbb{R}^3$ be the plane containing 0 perpendicular to the vector $(1,1,1)$. Using the standard norm, calculate the distance of the point $(1,2,3)$ to P.

Problem 4. Let $V = \mathbb{C}^4$ with its standard inner product. For $\theta \in \mathbb{R}$, let

$$v_\theta = \begin{pmatrix} 1 \\ e^{i\theta} \\ e^{2i\theta} \\ e^{3i\theta} \end{pmatrix} \in \mathbb{C}^4.$$

Find the canonical matrix of the orthogonal projection onto the subspace $\{v_\theta\}^\perp$.

Problem 5. Let $r \in \mathbb{R}$ and let $T \in \mathcal{L}(\mathbb{C}^2)$ be the linear map with canonical matrix

$$T = \begin{pmatrix} 1 & -1 \\ -1 & r \end{pmatrix}.$$

a) Find the eigenvalues of T.

b) Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of T.

c) Find a unitary matrix U such that UTU^* is diagonal.

Problem 6. Let A be the complex matrix given by:

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$$

a) Find the eigenvalues of A.

b) Find an orthonormal basis of eigenvectors of A.

c) Calculate $|A| = \sqrt{A^*A}$.

d) Calculate e^A.

Problem 7. Give an orthonormal basis for $\operatorname{null} T$, where $T \in \mathcal{L}(\mathbb{C}^4)$ is the map with canonical matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Problem 8. Describe the set of solutions $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ of the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

Problem 9. True or False? Check the box in front of the correct answer.

a) For any $n \geq 1$ and $A, B \in \mathbb{R}^{n\times n}$, one has $\det(A+B) = \det A + \det B$.
☐ True. ☐ False.

b) For any $r \in \mathbb{R}$, $n \geq 1$ and $A \in \mathbb{R}^{n\times n}$, one has $\det(rA) = r \det A$.
☐ True. ☐ False.

c) For any $n \geq 1$ and $A \in \mathbb{C}^{n\times n}$, one has $\operatorname{null} A = (\operatorname{ran} A)^\perp$.
☐ True. ☐ False.

d) The Gram-Schmidt process applied to an an orthonormal list of vectors reproduces that list unchanged.
☐ True. ☐ False.

e) Every unitary matrix is invertable.
☐ True. ☐ False.