LECTURE 15: PROPRIETIES OF (DOUBLE) SCHUBERT POLYNOMIALS

MIHAELA IFRIM

We recall from our previous lecture that:

$$\phi(C_{sp}) = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j) = \sum_{w \in S_n} \sigma_w(x, y) w.$$

We also defined $h_i(x) = 1 + xu_i$ where u_i are elements of the nilCoxeter algebra.

A slightly reformulation of the statement above is as follows:
Let $H_i(x) = h_{n-1}(x) \dots h_i(x)$. Recall that we showed last time that $[H_i(x), H_i(y)] = 0$. We defined

$$\sigma(x) = H_1(x_1) H_2(x_2) \dots H_{n-1}(x_{n-1})$$

and we showed last time

$$\phi(C_{sp}) = \sigma^{-1}(y)\sigma(x).$$

We can see (combining the facts above) that

$$\sigma_w(x, y) = \langle \sigma^{-1}(y)\sigma(x), w \rangle,$$

where $\langle v, w \rangle = \delta_{vw}$.

The single Schubert polynomial can be written as

$$\sigma_w(x) = \langle \sigma(x), w \rangle = \langle H_1(x_1) H_2(x_2) \dots H_{n-1}(x_{n-1}), w \rangle.$$

Example 0.1. Let us consider S_3. We compute

$$\begin{aligned} \phi(C_{sp}) &= h_2(x_1 - y_2) h_1(x_1 - y_1) h_2(x_2 - y_1) \\ &= (1 + (x_1 - y_2)u_2)(1 + (x_1 - y_1)u_1)(1 + (x_2 - y_1)u_2). \end{aligned}$$

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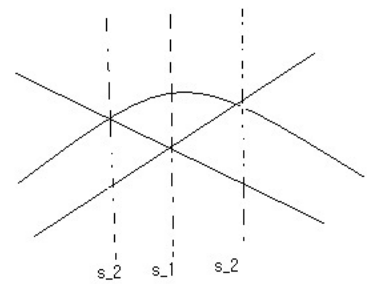
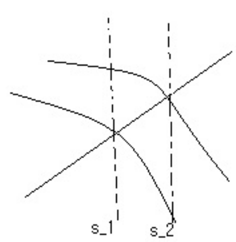
Using $\phi(C_{sp}) = \sum_{w \in S_n} \sigma_w(x,y) w$ we obtain:

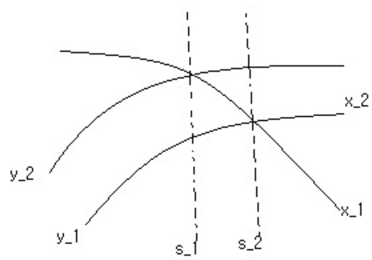


$\sigma_{s_2 s_1 s_2}(x,y) = (x_1-y_2)(x_1-y_1)(x_2-y_1)$

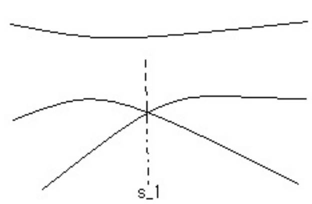
$\sigma_{s_1 s_2} = (x_1 - y_1)(x_2 - y_1)$



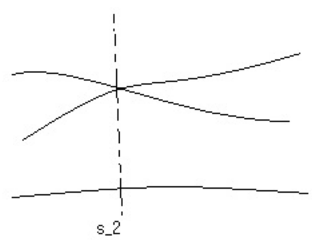
$\sigma_{s_2 s_1} = (x_1 - y_2)(x_1 - y_1)$



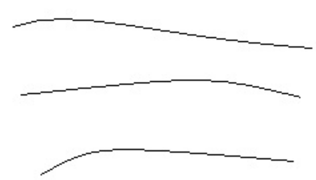
$\sigma_{s_1} = (x_1 - y_1)$



$\sigma_{s_2} = (x_1 - y_2) + (x_2 - y_1)$



$\sigma_1 = 1$



1. Symmetries

We observe that reflecting a configuration associated to a particular w along a vertical line yields a configuration asociated to w^{-1}.
The weight contributions of the crossings remain the same if we replace x_i by $-y_i$ and y_j by $-x_j$.

Corollary 1.1. *For any* $w \in S_n$

$$\sigma_{w-1}(x, y) = \sigma_w(-y, -x) = \varepsilon(w)\sigma_w(y, x).$$

Proof. This follows from the above explanation. □

Recall: $\partial_u \sigma_w = \begin{cases} \sigma_{wu^{-1}} & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u) \\ 0 & \text{else} \end{cases}$

Hence, $\partial_i \sigma_w = 0 \Leftrightarrow \ell(ws_i) > \ell(w) \Leftrightarrow \sigma_w$ is symmetric in x_i and x_{i+1}.

Corollary 1.2.

(1) $\sigma_w(x, y)$ *is symmetric in* x_i *and* $x_{i+1} \Leftrightarrow w(i) < w(i+1)$.
(2) $\sigma_w(x, y)$ *is symmetric in* y_i *and* $y_{i+1} \Leftrightarrow w^{-1}(i) < w^{-1}(i+1)$.

Remark 1.3. We have $\sigma_{s_i}(x, y) = x_1 + \cdots + x_i - y_1 - \cdots - y_i$. If r is the greatest descent of w i.e. r is greatest such that $w(r) > w(r+1)$ and if s is the greatest descent of w^{-1} then $\sigma_w(x, y)$ is a polynomal in $x_1, \ldots, x_r$, and in $y_1, \ldots, y_s$. (The proof of this last remark is left like an exercise. Hint: Note that $\sigma_w(x, y)$ is symmetric in $x_{r+1}, \ldots, x_n$ and it can also be checked that x_n does not appear.)

2. Stability

Denote by $i_n : S_n \to S_{n+1}$ the embedding that fixes $n+1$. The coresponding configuration is obtained by adjoing a strand on top that does not intersect any other strand.

Corollary 2.1. $w \in S_n$, $\sigma_w = \sigma_{i_n(w)}$.

More generally if $u \in S_n$, $v \in S_m$, define $u \times v = [u(1), \ldots, u(n), n+v(1), \ldots, n+v(m)] \in S_{n+m}$.

Corollary 2.2. *Let* $u \in S_n$, $v \in S_m$, $\sigma_{u \times v} = \sigma_u \cdot \sigma_{1^n \times v}$. *In particular we have the stability condition* $\sigma_u = \sigma_{u \times 1^s}$.

3. Stable Schubert polynomials or Stanley symmetric fuctions

Definition 3.1. $F_w(x) = \lim_{s \to \infty} \sigma_{1^s \times w}(x) = \langle H_1(x_1) H_1(x_2) \ldots, w \rangle$.

To justify the second equality in the definition, note that

$$\sigma_w(x) = \langle H_1^{n-1}(x_1) \ldots H_{n-1}^{n-1}(x_{n-1}), w \rangle,$$

where the top index in $H_i^{n-1}(x)$ indicates that the product over the h_j in H_i starts at $n-1$. Let $w = s_{a_1} \ldots s_{a_k}$ be a reduced expression of w. Replacing w by $1^s \times w$ we obtain

$$\begin{aligned} &\langle H_1^{n+s-1}(x_1) \ldots H_{n+s-1}^{n+s-1}(x_{n+s-1}), s_{a_{i+s}} \ldots s_{a_k+s} \rangle \\ =&\langle H_1^{n-1}(x_1) \ldots H_1^{n-1}(x_{s+1}) H_2^{n-1}(x_{s+2}) \ldots H_{n-1}^{n-1}(x_{n+s-1}), s_{a_1} \ldots s_{a_k} \rangle. \end{aligned}$$

If we take the limit when $s \to \infty$ we get $< H_1^{n-1}(x_1) H_1^{n-1}(x_2) \ldots, w >$.

Remark 3.2. Since $[H_i(x), H_i(y)] = 0$, $F_w(x)$ is symmetric in $x_1, x_2, \ldots$.

We recall that $\phi(C_{sp}) = \sigma^{-1}(y)\sigma(x)$. Setting $y = 0 \Rightarrow \sigma(x) = \sum_{w \in s_n} \sigma_w(x) w \quad (*)$

Setting $x = 0 \Rightarrow \sigma^{-1}(y) = \sum_{w \in S_n} \sigma_w(0, y) w = \sum_{w \in S_n} \varepsilon(w) \sigma_{w^{-1}}(y, 0) w = \sum_{w \in s_n} \sigma_{w^{-1}}(-y, 0) w$.
The second equality holds since we have symmetry.

Reformulating, we get:

$$\sigma^{-1}(y) = \sum_{w \in S_n} \sigma_{w^{-1}}(-y) w \quad (**)$$

Proposition 3.3. $\sigma_w(x, y) = \sum_{w = v^{-1}u, \ell(w) = \ell(u) + \ell(v)} \sigma_u(x) \sigma_v(-y).$

Proof. Just multiply (*) and (**) we get the desired result. □

Setting $w = w_0$ we obtain the Cauchy formula for Schubert polynomials:

Corollary 3.4.

$$\prod_{i+j \leq n} (x_i - y_j) = \sum_{w \in S_n} \sigma_w(x) \sigma_{ww_0}(-y).$$