LECTURE 2: COXETER GROUPS

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Definition 1.

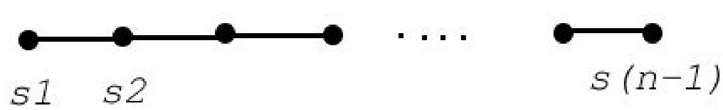
S is a set.
A matrix $m : S \times S \to \{1, 2, ..., \infty\}$ is called a Coxeter matrix if
$m(s, s\prime) = m(s\prime, s) \ \ \forall s, s\prime \epsilon S$
$m(s, s\prime) = 1 \iff s = s\prime$

Definition 2.

The Coxeter Group W with generators in S is the free group generated by S modulo the relations:
$(ss\prime)^{m(s,s\prime)} = e$ (where e is the identity element)
$\forall (s, s\prime)\epsilon S^2_{fin}$ where $S^2_{fin} = \{(s, s\prime)\epsilon S^2 \mid m(s, s\prime) \neq \infty\}$
(W, S) is called a Coxeter system with
generators: $s\epsilon S$
relations: $(ss\prime)^{m(s,s\prime)} = e \iff ss\prime ss\prime ... = s\prime ss\prime s... = m(s, s\prime)$

<u>Remark</u>: The Coxeter system (W, S) is uniquely determined by the Coxeter matrix m.

Example 1. <u>*Coxeter Group of type A_{n-1}*</u>:



relations:
$s_i^2 = e$ for $1 \leq i \leq n - 1$
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n - 2$
$s_i s_j = s_j s_i$ for $|i - j| > 1$
$\Longrightarrow$ Symmetric Group S_n

Example 2. <u>*Coxeter Group of type B_n*</u>:

relations:
$s_i^2 = e$ for $0 \leq i \leq n - 1$
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \leq i \leq n - 2$
$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$
$s_i s_j = s_j s_i$ for $|i - j| > 1$

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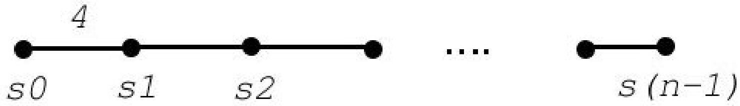
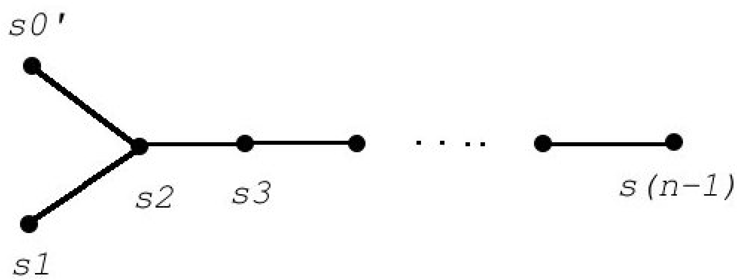


This is the group of signed permutations S_n^B
S_n^B is a group of bijections $w : \{\pm 1, ..., \pm n\} \to \{\pm 1, ..., \pm n\}$
write $w = [a_1, ..., a_n]$ where $w(i) = a_i$ and $w(-i) = -w(i)$
$s_0 = [-1, 2, ..., n]$
$s_i = [1, 2, ..., i-1, i+1, i, i+2, ..., n]$
Check relations!
Interpretation:
A deck of n cards where card j has j written on one side and $-j$ written on the other side. Elements of S_n^B can be identified with all possible arrangements of stacks of cards with orientation.

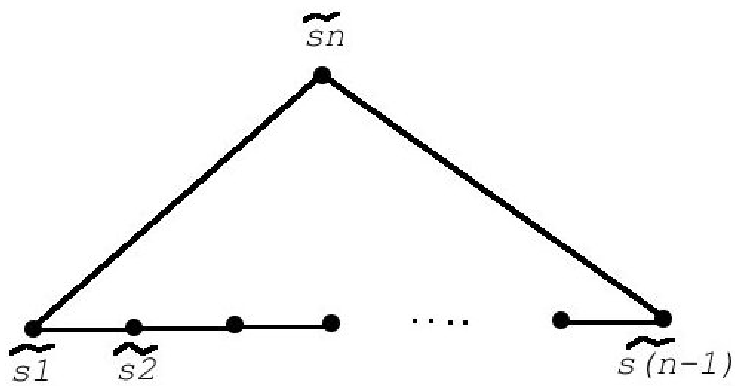
Example 3. *Coxeter Group of type D_n:*



relations:
$s_0\prime = s_0 s_1 s_0$ where s_0 is of type B_n
$(s_0\prime)^2 = e$
$s_0\prime s_1 = s_1 s_0\prime$
$s_0\prime s_2 s_0\prime = s_2 s_0\prime s_2$

[Note: $s_0^D = s_0^B s_1^B s_0^B and s_i^D = s_i^B$ for $i \geq 1$ where B and D represent the type of Coxeter Group.]
Interpretation:
S_n^D corresponds to all arrangements of stacks of n cards with orientation with an even number of cards turned over.

Example 4. *Coxeter Group of Affine type $\widetilde{A_{n-1}}$:*



affine permutation group $\widetilde{S_n}$
Affine permutations are permutations p of $\mathbb{Z}$ such that $p(j+n) = p(j) + n$ for $\forall j \epsilon \mathbb{Z}$
$\sum_{i+1}^{n} p(i) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$
generators: $\widetilde{s_i} = \prod_{j\epsilon\mathbb{Z}}(i+jn, i+1+jn)$ for $i = 1, ..., n$

Example 5. *Weyl Group of root systems:*

$\longrightarrow$ of importance in the theory of semisimple lie algebras
$\alpha \epsilon \mathbb{R}^d \setminus \{0\}$
reflections in the hyperplane orthogonal to α
$\sigma_\alpha(\alpha) = -\alpha$

Definition 3.

A finite set $\phi \subset \mathbb{R}^d \setminus \{0\}$ is a crystallographic root system if it spans $\mathbb{R}^d$ and $\forall \alpha, \beta \epsilon \phi$
(1) $\phi \cap \mathbb{R}_\alpha = \{\alpha, -\alpha\}$
(2) $\sigma_\alpha(\phi) = \phi$
(3) $\sigma_\alpha(\beta) = \beta + k\alpha$ for some $k \epsilon \mathbb{Z}$

$W = < \sigma_\alpha \mid \alpha \epsilon \phi >$