BRANCHING FORMULA FOR q-LITTLEWOOD-RICHARDSON COEFFICIENTS

A. SCHILLING, M. SHIMOZONO, AND D. E. WHITE

Abstract. A generalized inversion statistic is introduced on k-tuples of semistandard tableaux. It is shown that the cospin of a semistandard k-ribbon tableau is equal to the generalized inversion number of its k-quotient. This leads to a branching formula for the q-analogue of Littlewood-Richardson coefficients defined by Lascoux, Leclerc, and Thibon. This branching formula generalizes a recurrence of Garsia and Procesi involving Kostka-Foulkes polynomials.

1. Introduction

Let λ be a partition and $\mu^\bullet = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k-1)})$ a k-tuple of partitions. The Littlewood-Richardson (LR) coefficient

$$c^\lambda_{\mu^\bullet} = \langle s_\lambda \, , \, s_{\mu^{(0)}} s_{\mu^{(1)}} \cdots s_{\mu^{(k-1)}} \rangle, \tag{1.1}$$

is the coefficient of a Schur function in a product of Schur functions.

Lascoux, Leclerc, and Thibon [8] defined polynomials $c^\lambda_{\mu^\bullet}(q)$ such that

$$c^\lambda_{\mu^\bullet}(1) = c^\lambda_{\mu^\bullet} \, . \tag{1.2}$$

These q-analogues of LR coefficients are certain parabolic affine type A Kazhdan-Lusztig polynomials [9] and have nonnegative integer coefficients [7]. When each $\mu^{(i)}$ is a rectangle these polynomials are conjecturally equal to the generalized Kostka polynomials [6].

The main results of this paper are two branching rules (Theorem 5 and (3.8)) for the above q-LR coefficients, which generalize two recurrences of Garsia and Procesi [4] for the Kostka-Foulkes polynomials. The first branching rule follows immediately from a secondary result (Theorem 4), which gives a way to read the cospin of a k-ribbon tableau directly from its k-quotient (which is a k-tuple of semistandard tableaux) using a generalized inversion statistic.

Section 2 reviews the definition of the q-LR coefficients of [8]. Section 3 contains the new formula for the cospin and the branching rules. Section 4 contains the proof of Theorem 4. We conclude in Section 5 with some comments.

2. Definitions

2.1. **Cells, shapes, tableaux.** A *cell* is an ordered pair $s = (i, j)$ of positive integers. The English (matrix-style) convention is adopted here: (i, j) represents the position in the i-th row from the top and the j-th column from the left. Let $\mathrm{row}(s) = i$ and $\mathrm{col}(s) = j$ denote the row and column indices of the cell s. The (northwest to southeast) *diagonal* of the cell (i, j) is defined to be $\mathrm{diag}(i, j) = j - i$.

Second author partially supported by NSF DMS-9800941.

Write $s\downarrow$ for the cell $(i+1, j)$ just below $s = (i, j)$. Let $D(\lambda) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le j \le \lambda_i\}$ be the Ferrers diagram for the partition λ. For partitions μ and λ write $\mu \subset \lambda$ if $D(\mu) \subset D(\lambda)$. Young's lattice $\mathbb{Y}$ is the set of partitions under the partial order $\subset$. For a pair of partitions with $\mu \subset \lambda$, write λ/μ for the set $D(\lambda) - D(\mu)$; this is called a *skew shape*. For the skew shapes D and E, say that E *extends* D (or that D *is extended by* E) if there exist partitions $\nu \subset \mu \subset \lambda$ such that $D = \mu/\nu$ and $E = \lambda/\mu$. In this case let $D \cup E = \lambda/\nu$. A *horizontal strip* is a skew shape that has at most one cell in each column. A *standard tableau* of shape λ/μ is a saturated chain in $\mathbb{Y}$ from μ to λ. A *semistandard tableau* of shape λ/μ is a chain $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)} = \lambda$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip for all $1 \le i \le N$. The *weight* of this tableau is the sequence $(c_1, c_2, \dots, c_N)$ where c_i is the number of cells in $\lambda^{(i)}/\lambda^{(i-1)}$.

2.2. **Multipartitions and multitableaux.** Let k be a positive integer which shall be fixed from now on.

Denote by $\mathbb{Y}^k$ the k-fold direct product of the poset $\mathbb{Y}$, which is by definition the set of k-tuples of partitions $\lambda^\bullet = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k-1)})$ (call these k-*multipartitions*) with partial order $\mu^\bullet \subset \lambda^\bullet$ given by $\mu^{(i)} \subset \lambda^{(i)}$ for all $0 \le i \le k-1$. Let $\mu^\bullet, \lambda^\bullet \in \mathbb{Y}^k$ with $\mu^\bullet \subset \lambda^\bullet$. Write $\lambda^\bullet/\mu^\bullet = (\lambda^{(0)}/\mu^{(0)}, \dots, \lambda^{(k-1)}/\mu^{(k-1)})$, a k-tuple of skew shapes. A *standard k-multitableau* of shape $\lambda^\bullet/\mu^\bullet$ is a saturated chain in $\mathbb{Y}^k$ from $\mu^\bullet$ to $\lambda^\bullet$.

A cell in a k-multipartition $\lambda^\bullet$ is a triple $s = (i, j, p) \in \mathbb{Z}^2 \times \{0, 1, \dots, k-1\}$ such that $(i, j) \in \lambda^{(p)}$. Write $\mathrm{pos}(s)$ for the index p and $\mathrm{row}(s), \mathrm{col}(s), \mathrm{diag}(s)$ for the corresponding notions applied to (i, j). Define the partial order $\preceq$ on elements of $\mathbb{Z}^2 \times \{0, 1, \dots, k-1\}$ by $s \prec s'$ if $\mathrm{diag}(s) < \mathrm{diag}(s')$, or if $\mathrm{diag}(s) = \mathrm{diag}(s')$ and $\mathrm{pos}(s) < \mathrm{pos}(s')$. This partial order appears in a realization of the crystal graph of irreducible integrable highest weight modules of affine type A and level k [1].

Say that $\lambda^\bullet/\mu^\bullet$ is a *horizontal k-multistrip* if it is a k-tuple of horizontal strips. Suppose this is the case. Then the cells of $\lambda^\bullet/\mu^\bullet$ are linearly ordered under $\preceq$. Define the *standardization* of $\lambda^\bullet/\mu^\bullet$ to be the unique saturated chain from $\mu^\bullet$ to $\lambda^\bullet$ in $\mathbb{Y}^k$ in which the cells of $\lambda^\bullet/\mu^\bullet$ are added in $\preceq$-increasing order. A *semistandard k-multitableau* of shape $\lambda^\bullet/\mu^\bullet$ is a chain from $\mu^\bullet$ to $\lambda^\bullet$ in $\mathbb{Y}^k$ such that each k-tuple in the chain is obtained from the previous one by extension by a horizontal k-multistrip. A semistandard k-multitableau T can be viewed as a k-tuple $T^\cdot = (T^{(0)}, T^{(1)}, \dots, T^{(k-1)})$ of semistandard Young tableaux. The standardization of a semistandard k-multitableau is defined by joining the standardizations of the above horizontal k-multistrips.

2.3. **k-ribbons and tableaux.** A *k-ribbon* is a connected skew shape consisting of k cells, at most one on each diagonal. From now on, all ribbons shall be k-ribbons. The *head* (written $\mathrm{head}(R)$) of the ribbon R is its northeastmost cell. The *tail* (written $\mathrm{tail}(R)$) of R is its southwestmost cell. Define the *spin* of the ribbon R by $\mathrm{spin}(R) = \mathrm{row}(\mathrm{tail}(R)) - \mathrm{row}(\mathrm{head}(R))$. Clearly for a k-ribbon R, $\mathrm{spin}(R) \in \{0, 1, 2, \dots, k-1\}$.

Define the relation $\mu \lessdot_k \lambda$ on $\mathbb{Y}$ to mean that $\mu \subset \lambda$ and the skew shape $R = \lambda/\mu$ is a k-ribbon. In this situation R is said to be *μ-addable* and *λ-removable*. The relation $\lessdot_k$ is the covering relation for a partial order $\le_k$ on $\mathbb{Y}$. Each component of the poset $(\mathbb{Y}, \le_k)$ has a unique minimum. The $\le_k$-minima are called k-cores. The k-ribbon lattice $\mathcal{R}_k$ is the component of the empty partition $\varnothing$ in $(\mathbb{Y}, \le_k)$. If

$\mu \leq_k \lambda$ are partitions then a *standard k-ribbon tableau* of shape λ/μ is a saturated $\leq_k$-chain from μ to λ. The spin of a standard k-ribbon tableau is the sum of the spins of the k-ribbons formed by the successive differences of partitions in the chain.

Let $\mu \leq_k \lambda$ be partitions. The skew shape λ/μ is a *horizontal k-ribbon strip* if it admits a standard k-ribbon tableau $\mu = \lambda^{(0)} \lessdot_k \lambda^{(1)} \lessdot_k \cdots \lessdot_k \lambda^{(m)} = \lambda$ such that, writing $R_i = \lambda^{(i)}/\lambda^{(i-1)}$, $\mathrm{head}(R_i)$ is the topmost cell in its column within λ/μ, for all $1 \leq i \leq m$. In this case there is a unique such tableau with the property that the heads of the ribbons R_i appear in order from left to right; it is called the standardization of the horizontal k-ribbon strip λ/μ. The spin of a horizontal k-ribbon strip λ/μ is the spin of its standardization.

A *semistandard k-ribbon tableau* T of shape λ/μ is a chain $\mu = \lambda^{(0)} \leq_k \lambda^{(1)} \leq_k \ldots \leq_k \lambda^{(M)} = \lambda$ such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal k-ribbon strip for $1 \leq i \leq M$. Its weight is the sequence $\mathrm{weight}(T) = (c_1, c_2, \ldots, c_M)$ where c_i is the number of k-ribbons in the standardization of $\lambda^{(i)}/\lambda^{(i-1)}$. The standardization of a semistandard k-ribbon tableau is the standard k-ribbon tableau obtained by joining together the standardizations of the horizontal ribbon strips. The spin of a semistandard k-ribbon tableau is the spin of its standardization. Write $\mathrm{Tab}_k(\lambda)$ (resp. $\mathrm{STab}_k(\lambda)$) for the set of semistandard (resp. standard) k-ribbon tableaux of shape λ.

2.4. **k-quotient.** Littlewood's k-quotient map gives a poset isomorphism $\mathrm{quot}_k : \mathcal{R}_k \to \mathbb{Y}^k$ (see [3]). We use the following normalization of the k-quotient. Let λ be a partition. By appending zeros if necessary it may be assumed that the number of parts of λ is a multiple mk of k. Define the staircase partition $\rho^{(r)} = (r-1, r-2, \ldots, 1, 0) \in \mathbb{Z}^r$. It can be shown that λ has empty k-core (that is, $\lambda \in \mathcal{R}_k$) if and only if $\lambda + \rho^{(mk)}$ has exactly m parts that are congruent to i mod k for each $0 \leq i \leq k-1$. In that case, define the k-quotient $\mathrm{quot}_k(\lambda)$ of λ to be the k-tuple of partitions $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k-1)})$ where the partition $k(\lambda^{(i)} + \rho^{(m)}) + i$ coincides with the partition obtained by selecting the parts of $\lambda + \rho^{(mk)}$ that are congruent to i modulo k. This definition is independent of m.

Example 1. Let $k = 5$ and $\lambda = (14, 12, 11, 10, 10, 10, 10, 9, 6, 4, 3, 3, 2, 1)$. Then $\mathrm{quot}_5(\lambda) = ((3, 3), (2, 2, 1), (), (3, 3, 3), (1))$.

Remark 2. Let $\mu, \lambda \in \mathcal{R}_k$. One has the following dictionary between k-ribbon tableaux and k-multitableaux.

(1) $\mu \leq_k \lambda$ if and only if $\mathrm{quot}_k(\mu) \subset \mathrm{quot}_k(\lambda)$.
(2) $\mu \lessdot_k \lambda$ if and only if $\mathrm{quot}_k(\lambda)$ is obtained by adjoining a cell to one of the partitions in $\mathrm{quot}_k(\mu)$.
(3) The map $\mathrm{quot}_k : \mathcal{R}_k \to \mathbb{Y}^k$ induces a bijection from the set of standard k-ribbon tableaux of shape μ, to the set of standard k-multitableaux of shape $\mathrm{quot}_k(\mu)$.
(4) λ/μ is a horizontal k-ribbon strip if and only if $\mathrm{quot}_k(\lambda)/\mathrm{quot}_k(\mu)$ is a horizontal k-multistrip.
(5) Let std denote all the various standardization maps. If λ/μ is a horizontal k-ribbon strip, then $\mathrm{quot}_k(\mathrm{std}(\lambda/\mu)) = \mathrm{std}(\mathrm{quot}_k(\lambda)/\mathrm{quot}_k(\mu))$.
(6) The map quot_k induces a weight-preserving bijection

$$\mathrm{quot}_k : \mathrm{Tab}_k(\mu) \to \prod_{i=0}^{k-1} \mathrm{Tab}_1(\mu^{(i)}) \tag{2.1}$$

from the semistandard k-ribbon tableaux of shape μ, to the semistandard k-multitableaux of shape $\mathrm{quot}_k(\mu)$ (which are the same thing as k-tuples of semistandard Young tableaux whose shapes are given by $\mathrm{quot}_k(\mu)$).

The above dictionary is missing a statistic on k-multitableaux that corresponds to spin on k-ribbon tableaux. This omission is rectified in section 3.

2.5. **q-LR coefficients.** Let $\mu \in \mathcal{R}_k$. The cospin of $T \in \mathrm{Tab}_k(\mu)$ is defined by

$$\begin{aligned} \mathrm{maxspin}(\mu) &= \max \{ \mathrm{spin}(S) \mid S \in \mathrm{Tab}_k(\mu)\} \\ \mathrm{cospin}(T) &= \frac{1}{2}(\mathrm{maxspin}(\mu) - \mathrm{spin}(T)). \end{aligned} \tag{2.2}$$

It is not hard to show that cospin is always an integer. Following [8] define

$$\widetilde{G}_\mu[X;q] = \sum_{T \in \mathrm{Tab}_k(\mu)} q^{\mathrm{cospin}(T)} x^{\mathrm{weight}(T)}. \tag{2.3}$$

where $X = (x_1, x_2, \dots)$. The series $\widetilde{G}_\mu[X;q]$ is a symmetric function in X [8]. Let $\mu^\bullet = \mathrm{quot}_k(\mu)$. Define the polynomials $c^\lambda_{\mu^\bullet}(q)$ by

$$\widetilde{G}_\mu[X;q] = \sum_\lambda c^\lambda_{\mu^\bullet}(q) s_\lambda[X]. \tag{2.4}$$

Setting $q = 1$ in (2.3) one obtains

$$\widetilde{G}_\mu[x;1] = s_{\mu^{(0)}}[X] s_{\mu^{(1)}}[X] \cdots s_{\mu^{(k-1)}}[X] \tag{2.5}$$

due to Remark 2 point 6. Setting $q = 1$ in (2.4) one obtains (1.2).

3. Cospin formula and branching rule

In this section we indicate how to compute the cospin of an k-ribbon tableau directly in terms of its k-quotient.

Let $\mathrm{Bot}(\mu) \subset D(\mu)$ be the subset of cells that are bottommost in their columns, and $\mathrm{Corn}(\mu) \subset \mathrm{Bot}(\mu)$ the set of corner cells in $D(\mu)$. For a k-multipartition $\mu^\bullet$ define the subsets of $\mathbb{Z}^2 \times \{0, 1, \dots, k-1\}$ given by the disjoint unions $\mathrm{Bot}(\mu^\bullet) = \coprod_{i=0}^{k-1} \mathrm{Bot}(\mu^{(i)})$ and $\mathrm{Corn}(\mu^\bullet) = \coprod_{i=0}^{k-1} \mathrm{Corn}(\mu^{(i)})$.

3.1. **Inversions of a ribbon tableau.** Let T be a standard k-ribbon tableau of shape μ. For $s \in \mathrm{quot}_k(\mu)$ by abuse of notation write $T(s)$ for the value of the cell s in $\mathrm{quot}_k(T)$, which is the index i such that s is in the i-th multipartition in the chain T but not in the $(i-1)$-st.

Given cells $s, t \in \mathrm{quot}_k(\mu)$, say that (s,t) is an *inversion* of T provided the following conditions hold:

1. $\mathrm{diag}(s) = \mathrm{diag}(t)$ and $\mathrm{pos}(s) < \mathrm{pos}(t)$, or $\mathrm{diag}(s) = \mathrm{diag}(t) - 1$ and $\mathrm{pos}(s) > \mathrm{pos}(t)$. Note that in either case $s \prec t$.
2. $\mathrm{row}(s) \le \mathrm{row}(t)$.
3. $T(t) < T(s) < T(t{\downarrow})$ (where $T(t{\downarrow}) = \infty$ if $t{\downarrow} \notin \mathrm{quot}_k(\mu)$).

Write $\mathrm{inv}(T)$ for the number of inversions of T. When $\mathrm{quot}_k(\mu) = ((1), \dots, (1))$ then the standard multitableau $\mathrm{quot}_k(T)$ can be identified with a permutation of the set $\{1, 2, \dots, k\}$ and $\mathrm{inv}(T)$ is the usual inversion number. The number of inversions of a semistandard k-ribbon tableau is the number of inversions of its standardization.

There is an obviously equivalent recursive way to compute $\mathrm{inv}(T)$. Let $s \in \mathrm{Corn}(\mu^\bullet)$. Write $\mathrm{Inv}(s,\mu^\bullet)$ for the set of cells $t \in \mathrm{Bot}(\mu^\bullet) - s$ such that the above properties (1) and (2) hold. Write $\mathrm{inv}(s,\mu^\bullet) = |\mathrm{Inv}(s,\mu^\bullet)|$.

Recall that T has shape $\mu \in \mathcal{R}_k$. Write $\mu^\bullet = \mathrm{quot}_k(\mu)$. Let R be the k-ribbon that contains the largest value in T. Let $s \in \mathrm{Corn}(\mu^\bullet)$ be given by $\mathrm{quot}_k(R) = s$; it contains the largest value in $\mathrm{quot}_k(T)$. Then

$$\mathrm{inv}(T) = \mathrm{inv}(s,\mu^\bullet) + \mathrm{inv}(T - R). \tag{3.1}$$

Example 3. Let k, μ be as in Example 1. $\mathrm{Bot}(\mathrm{quot}_k(\mu))$ consists of 9 cells $s_1 \prec s_2 \prec \ldots \prec s_9$. Below, i is written instead of s_i.

The list of inversions (s,t) for $s \in \mathrm{Corn}(\mathrm{quot}_k(\mu))$ is $\{(1,2),(6,7),(6,9),(8,9)\}$.

3.2. **Cospin is the number of inversions.**

Theorem 4. *The number of inversions of a semistandard k-ribbon tableau is equal to its cospin.*

Theorem (4) is proved in section 4. Given $\mu^\bullet \in \mathbb{Y}^k$, define

$$W(\mu^\bullet;q) = \sum_\lambda f^\lambda c^\lambda_{\mu^\bullet}(q) \tag{3.2}$$

where f^λ is the number of standard tableaux of shape λ. Equivalently, letting $\mu \in \mathcal{R}_k$ be such that $\mathrm{quot}_k(\mu) = \mu^\bullet$,

$$W(\mu^\bullet;q) = \sum_{T \in \mathrm{STab}_k(\mu)} q^{\mathrm{cospin}(T)}, \tag{3.3}$$

since $W(\mu^\bullet;q)$ (resp. f^λ) is the coefficient of $x_1 x_2 \cdots x_r$ in $\widetilde{G}_\mu(x;q)$ (resp. $s_\lambda(x)$) where r is the number of cells in $\mu^\bullet$.

3.3. **Generalized Garsia-Procesi recurrence.** The following recurrence is an immediate consequence of Theorem 4 and (3.1).

Theorem 5.

$$W(\mu^\bullet;q) = \sum_{s \in \mathrm{Corn}(\mu^\bullet)} q^{\mathrm{inv}(s,\mu^\bullet)} W(\mu^\bullet - s;q).$$

Consider the special case that $\mu^\bullet$ consists of a sequence of single-rowed shapes $\mu^{(i-1)} = (\alpha_i)$ for $1 \leq i \leq k$ where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence of nonnegative integers. In this case it is known [8] that the q-LR coefficient $c^\lambda_{\mu^\bullet}(q) = \widetilde{K}_{\lambda\alpha^+}(q)$ is the cocharge Kostka-Foulkes polynomial where α^+ is the partition obtained by sorting α. Let s be the unique corner cell of $\mu^{(i)}$ (assuming $\alpha_i > 0$). Then $\mathrm{inv}(s,\mu^\bullet)$ is the number of indices j such that either $\alpha_j > \alpha_i$, or $\alpha_j = \alpha_i$ and $j > i$. In other words, to find the power of q associated with the corner cells in this case, one can label the longest parts of α from right to left with the numbers $0, 1, 2, \ldots$, then the next longest parts of α from left to right, and continue in this way. One can see that when α is a partition this is the recurrence [4, (1.7)] of Garsia and Procesi, whose proof uses combinatorial, algebraic and geometric techniques. K. Killpatrick has a combinatorial (but not entirely bijective) proof of this recurrence, working with words and charge [5].

Example 6. Let $k = 7$ and $\mu^\bullet = ((2), (1), (), (2), (1), (3), (2))$. Then the associated powers of q are $(3, 5, 6, 2, 4, 0, 1)$, although the 6 doesn't appear in the formula since $()$ has no corner cell.

3.4. **Another branching formula.** Let X and Y be alphabets. Using plethystic notation we expand $\widetilde{G}_\mu[X+Y;q]$ in two ways, obtaining a more general recurrence for $c^\lambda_{\mu^\bullet}(q)$. The generating function $\widetilde{G}_D[X;q]$ may be defined for any skew shape $D = \mu/\nu$ such that $\nu \le_k \mu$. Let

$$a(\mu,\nu) = \frac{1}{2}(\mathrm{maxspin}(\mu) - \mathrm{maxspin}(\nu) - \mathrm{maxspin}(\mu/\nu)).$$

Directly from the definitions and (2.3) one has the expansion

$$\widetilde{G}_\mu[X+Y;q] = \sum_{\nu \le_k \mu} q^{a(\mu,\nu)} \widetilde{G}_\nu[X;q] \widetilde{G}_{\mu/\nu}[Y;q]. \tag{3.4}$$

If spin is used in the generating function $\widetilde{G}_\mu[X;q]$ instead of cospin then one may omit the power of q. Let $s^\perp_\alpha[X]$ be the adjoint operator of multiplication by $s_\alpha[X]$, with respect to the scalar product on symmetric functions with respect to which the Schur functions are orthonormal. From the coproduct expansion (3.4) one has

$$s^\perp_\alpha[X]\widetilde{G}_\mu[X;q] = \sum_{\nu \le_k \mu} q^{a(\mu,\nu)} \widetilde{G}_\nu[X;q] \langle \widetilde{G}_{\mu/\nu}[Y;q]\,,\, s_\alpha[Y]\rangle. \tag{3.5}$$

Suppose $\alpha = (r)$. Then the scalar product on the right hand side of (3.5) is nonzero if and only if μ/ν is a horizontal ribbon strip with r ribbons. Assume this is the case. Then the value of the scalar product is 1 and

$$h^\perp_r[X]\widetilde{G}_\mu[X;q] = \sum_{\nu \le_k \mu} q^{a(\mu,\nu)} \widetilde{G}_\nu[X;q] \tag{3.6}$$

where μ/ν is a horizontal ribbon strip with r ribbons. The constant $a(\mu,\nu)$ can be computed as follows. For any skew k-ribbon tableau S one may define $\mathrm{inv}(S)$ using (3.1). For any $\nu \le_k \mu$, if S is a k-ribbon tableau such that $\mathrm{spin}(S) = \mathrm{maxspin}(\mu/\nu)$, it follows by summing (4.1) over the ribbons in S, that $a(\mu,\nu) = \mathrm{inv}(S)$. In the current case, there is a unique such tableau S, namely, the standardization of the horizontal ribbon strip μ/ν. In this case define $\mathrm{inv}(\mu/\nu) = \mathrm{inv}(S)$. Then

$$h^\perp_r[X]\widetilde{G}_\mu[X;q] = \sum_{\nu \le_k \mu} q^{\mathrm{inv}(\mu/\nu)} \widetilde{G}_\nu[X;q]. \tag{3.7}$$

where μ/ν is a horizontal ribbon strip with r ribbons. Taking the coefficient of s_ρ where ρ is a partition of size r less than that of $\mu^\bullet = \mathrm{quot}_k(\mu)$, one has

$$\sum_{\lambda \supset \rho} c^\lambda_{\mu^\bullet}(q) = \sum_{\nu \le_k \mu} q^{\mathrm{inv}(\mu/\nu)} c^\rho_{\nu^\bullet}(q), \tag{3.8}$$

where λ runs over the partitions such that λ/ρ is a horizontal strip of size r and ν is such that μ/ν is a horizontal k-ribbon strip containing r ribbons, and $\nu^\bullet = \mathrm{quot}_k(\nu)$.

Taking $\alpha = (1^r)$ instead of $\alpha = (r)$, one obtains a similar formula, which specializes to [4, Prop. 5.3] when all the partitions in $\mu^\bullet$ are single rows.

4. Proof of Theorem 4

4.1. **Starting the proof of Theorem 4.** Let T be a standard k-ribbon tableau of shape μ, R the k-ribbon that contains the largest value of T, and $T-R$ the standard k-ribbon tableau obtained by removing R from T. Write $s = \mathrm{quot}_k(R)$. By induction and the definitions we have

$$\begin{aligned} 2\,\mathrm{cospin}(T) &= \mathrm{maxspin}(\mu) - \mathrm{spin}(T) = \mathrm{maxspin}(\mu) - \mathrm{spin}(T-R) - \mathrm{spin}(R) \\ &= \mathrm{maxspin}(\mu) - \mathrm{maxspin}(\mu - R) + 2\,\mathrm{inv}(T-R) - \mathrm{spin}(R) \\ &= \mathrm{maxspin}(\mu) - \mathrm{maxspin}(\mu - R) + 2\,(\mathrm{inv}(T) - \mathrm{inv}(s, \mathrm{quot}_k(\mu)) \\ &\quad - \mathrm{spin}(R). \end{aligned}$$

Hence the proof of Theorem 4 is reduced to that of

$$\mathrm{maxspin}(\mu) - \mathrm{maxspin}(\mu - R) - \mathrm{spin}(R) = 2\,\mathrm{inv}(s, \mathrm{quot}_k(\mu)). \tag{4.1}$$

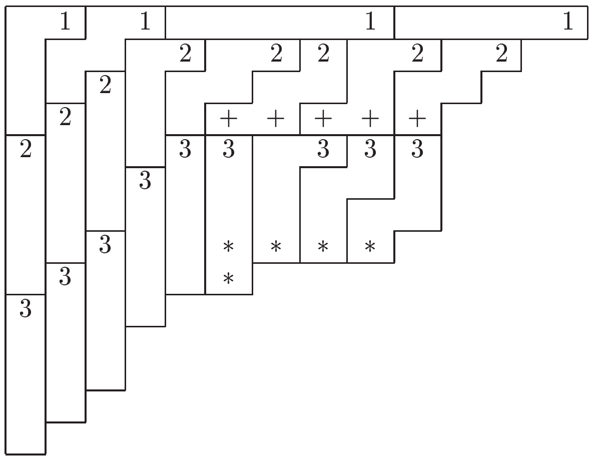
4.2. **Maxspin computation.** To prove (4.1) the quantity $\mathrm{maxspin}(\mu)$ must be computed. Given a partition ν with at most m parts, let $Y_m(\nu)$ be the unique semistandard tableau of shape ν and weight $(\nu_m, \nu_{m-1}, \dots, \nu_1)$. Usually m is suppressed in the notation. Write $\mathrm{quot}_k(\mu) = \mu^\bullet$. Fix a positive integer m such that $\mu^{(i)}$ has at most m parts for all i. Let Y_μ be the semistandard ribbon tableau of shape μ such that $\mathrm{quot}_k(Y_\mu) = (Y(\mu^{(0)}), Y(\mu^{(1)}), \dots, Y(\mu^{(k-1)}))$.

Lemma 7. $\mathrm{spin}(Y_\mu) = \mathrm{maxspin}(\mu)$.

Example 8. In our running example with $k = 5$ and $m = 3$, $\mathrm{quot}_k(Y_\mu)$ is given by

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ 3 & 3 & 3 \\ \hline \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \end{array} \qquad \varnothing \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ \hline \end{array} \qquad \boxed{3}$$

The ribbon tableau Y_μ is given by



Proof. (Of Lemma 7) Let m be a sufficiently large positive integer. Let $T \in \mathrm{Tab}_k(\mu)$ have maximum value at most m. Let Y_μ be defined with respect to m. Observe that $\mathrm{spin}(Y_\mu)$ is independent of m. The proof proceeds by induction on the antidominance order on $\mathbb{N}^m$, defined by $(\alpha_1, \dots, \alpha_m) \trianglerighteq^* (\beta_1, \dots, \beta_m)$ if and only if $\alpha_i + \alpha_{i+1} + \cdots + \alpha_m \geq \beta_i + \beta_{i+1} + \cdots + \beta_m$ for all $1 \leq i \leq m$. Suppose $T \neq Y_\mu$. Among all semistandard ribbon tableaux of shape μ with maximum value at most m, Y_μ has the most antidominant weight, and is the unique such tableau. This holds because the k-quotient map preserves weight, and because $Y(\nu)$ is the unique semistandard tableau of most antidominant weight, among semistandard tableaux of shape ν with maximum value at most m. By induction on antidominance it suffices to find $T' \in \mathrm{Tab}_k(\mu)$ with maximum value at most m, such that $\mathrm{spin}(T) \le \mathrm{spin}(T')$ and $\mathrm{weight}(T') \rhd^* \mathrm{weight}(T)$. Since $T \ne Y_\mu$ there is a value $r < m$ and a cell $s \in \mathrm{quot}_k(\mu)$ such that $T(s) = r$ and $T(s\downarrow) \ne r+1$, where recall that $T(s\downarrow) = \infty$ if $s\downarrow \notin \mathrm{quot}_k(\mu)$. Fix any such value r and choose s to be the unique $\preceq$-maximum cell with the above properties. Let $T' \in \mathrm{Tab}_k(\mu)$ be such that $\mathrm{quot}_k(T)$ and $\mathrm{quot}_k(T')$ differ only by changing the value in the cell s from r to $r+1$. Then $\mathrm{weight}(T') \rhd^* \mathrm{weight}(T)$. It suffices to show $\mathrm{spin}(T) \le \mathrm{spin}(T')$.

Let R be the ribbon in T such that $\mathrm{quot}_k(R) = s$. In passing from T to T', the ribbons R' in T that move are those that share a diagonal with R, and whose quotients s' either 1) contain the value r and satisfy $s \prec s'$ (in which case $\mathrm{tail}(R')$ moves northwest), or 2) contain the value $r+1$ and $s' \prec s$ (in which case $\mathrm{head}(R')$ moves northwest). More specifically, in passing from T to T':

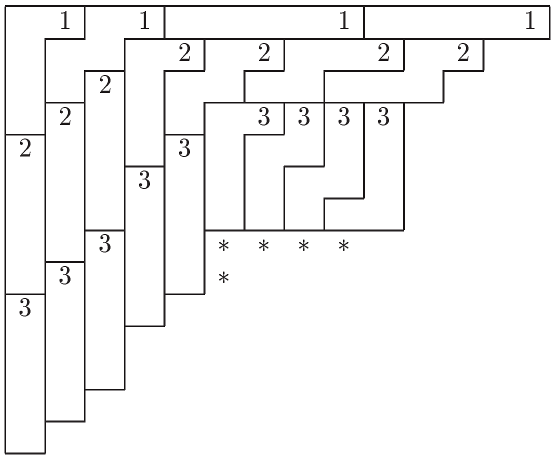
(1) Spin decreases by 2 for a) every r with the same diagonal and greater "pos" than s and for b) every r on the next larger diagonal and smaller pos.
(2) Spin increases by 2 for a) every r+1 in the next smaller diagonal and greater pos and for b) every r+1 in the same diagonal but with smaller pos.

However the r's as in 1a) must have an r+1 as in 2a) just below them and the r's as in 1b) must have an r+1 below them as in 2b), due to the $\preceq$-maximality of the cell s. This shows that the spin increases or stays the same. $\square$

4.3. **Proof of** (4.1)**.** The notation of section 4.1 is used here. By Lemma 7 the verification of (4.1) requires the computation of $\mathrm{spin}(Y_\mu) - \mathrm{spin}(Y_{\mu-R})$.

Remark 9. Observe that $\mathrm{quot}_k(Y_{\mu-R})$ is obtained from $\mathrm{quot}_k(Y_\mu)$ by a reverse row insertion at the cell s. The bumping path of this insertion goes straight upwards along the column of s, ejecting the letter $m+1-\mathrm{row}(s)$.

Example 10. Let $s = s_6$ as in Example 3. In the picture of Y_μ in Example 8, the cells of R are indicated by $*$. Here is $Y_{\mu-R}$, with the ribbon R indicated by the symbol $*$.



Here is $\mathrm{quot}_k(Y_{\mu-R})$.

$$\begin{array}{ccc} 2 & 2 & 2 \\ 3 & 3 & 3 \end{array} \qquad \begin{array}{cc} 1 & 3 \\ 2 & \\ 3 & \end{array} \qquad \varnothing \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array} \qquad \begin{array}{c} 3 \end{array}$$

We use the following fact [10].

Theorem 11. *The Stanton-White k-ribbon row insertion algorithm corresponds under the k-quotient map, to a k-fold product of ordinary Schensted row insertion algorithms.*

Consequently the above reverse row insertion on the quotient, is the image under the k-quotient map, of the reverse Stanton-White row insertion on Y_μ starting at the ribbon R. The change in spin under this operation is now calculated.

Write $Y = Y_\mu$ and $Y' = Y_{\mu-R}$. Suppose first that $\mathrm{row}(s) > 1$. Let H (resp. H') be the horizontal ribbon strip in Y (resp. Y') of value m. Observe that $R \subset H$ since $\mathrm{quot}_k(R) = s \in \mathrm{Corn}(\mu^\bullet) \subset \mathrm{Bot}(\mu^\bullet) = \mathrm{quot}_k(H)$ where $\mu^\bullet = \mathrm{quot}_k(\mu)$. It follows from Theorem 11 that the skew shape R extends H' and that there is a k-ribbon R' such that H extends R' and with $R' \cup H = H' \cup R$. We will show that

$$\mathrm{spin}(H) - \mathrm{spin}(R) = \mathrm{spin}(H') - \mathrm{spin}(R'). \tag{4.2}$$

Example 12. For $Y = Y_\mu$ as in Example 8 the cells of R (resp. R') are indicated by the symbol $*$ (resp. $+$). See $Y' = Y_{\mu-R}$ in Example 10. Then (4.2) gives $33 - 1 = 32 - 0$.

Equation (4.2) allows a reduction to the case $\mathrm{row}(s) = 1$. We have

$$\begin{aligned} &\mathrm{spin}(Y) - \mathrm{spin}(Y') - \mathrm{spin}(R) \\ &= \mathrm{spin}(Y-H) + \mathrm{spin}(H) - \mathrm{spin}(Y'-H') - \mathrm{spin}(H') - \mathrm{spin}(R) \\ &= \mathrm{spin}(Y-H) - \mathrm{spin}(Y'-H') - \mathrm{spin}(R'). \end{aligned} \tag{4.3}$$

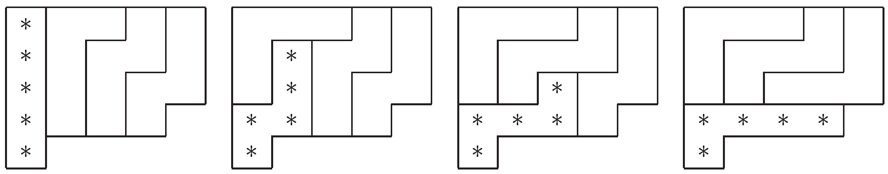
using (4.2). Let $\nu = \mu - H$, $s' = \mathrm{quot}_k(R')$, and $\nu' = \nu - s'$. Observe that $s' \!\downarrow = s$ and that $\mathrm{quot}_k(\nu) = \mathrm{quot}_k(\mu) - \mathrm{Bot}(\mathrm{quot}_k(\mu))$. Using $m-1$ instead of m, one sees also that $Z = Y_\nu$ and $Z' = Y_{\nu'}$ satisfy $Z = Y - H$ and $Z' = Y' - H'$. By induction (4.1) holds for Z and Z' so that the right hand side of (4.3) is equal to $2\,\mathrm{inv}(s', \mathrm{quot}_k(\nu))$. But there is a bijection $\mathrm{Inv}(s', \mathrm{quot}_k(\nu)) \to \mathrm{Inv}(s, \mathrm{quot}_k(\mu))$ given by $t \mapsto t\!\downarrow$. From this (4.1) follows.

To complete the reduction to the case $\mathrm{row}(s) = 1$, let us prove (4.2). The Stanton-White algorithm computes the pair (H', R') from (H, R) as follows. There are three phases.

Phase I: This phase computes the difference of the spins of H and $H - R$. Let $R_1, \dots, R_M$ be the ribbons of $\mathrm{std}(H)$ indexed from left to right, and R_ℓ through $R_{\ell+c}$ those that intersect R. Since $\mathrm{row}(s) > 1$, a letter m is bumped in the reverse row insertion of Remark 9. By Theorem 11 it follows that a ribbon of value m is bumped in the Stanton-White reverse row insertion. But this is equivalent to the condition $\mathrm{tail}(R_\ell) = \mathrm{tail}(R)$. The ribbons R_1 through $R_{\ell-1}$ and $R_{\ell+c+1}$ through R_M are also ribbons in $\mathrm{std}(H - R)$. The ribbons R_ℓ through $R_{\ell+c}$ in $\mathrm{std}(H)$ are replaced by ribbons $R'_{\ell+1}, \dots, R'_{c+\ell}$ in $\mathrm{std}(H - R)$ as follows. The ribbon R_ℓ is declared to be the active ribbon. It is switched to the southeast, past the ribbons $R_{\ell+1}$ through $R_{\ell+c}$, which become $R'_{\ell+1}$ through $R'_{\ell+c}$. At the end the active ribbon is R. This switching process (which is equivalent to the "bumpin" of [10]) has the following properties: $\text{head}(R'_i) = \text{head}(R_i)$ for $\ell < i \le \ell + c$ and $\text{tail}(R'_i)$ is obtained from $\text{tail}(R_i)$ by moving one cell northwest. It follows that

$$\begin{aligned} \text{spin}(R_\ell) &= \text{spin}(R) + c \\ \text{spin}(H) - \text{spin}(R_\ell) &= \text{spin}(H - R) + c. \end{aligned} \tag{4.4}$$

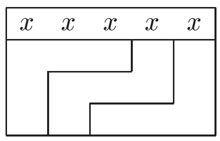
Example 13. Here H is the horizontal ribbon strip in T of value 3, $\ell = 6$ (because $s = s_6 = \text{quot}_k(R)$) and $c = 3$. We start with a picture of the part of H consisting of R_ℓ through $R_{\ell+c}$. The active ribbon (which at the beginning is R_ℓ) is marked with the symbol $*$.



The last picture is the standardization of the part of $H - R$ given by $R'_{\ell+1}$ through $R'_{c+\ell}$, extended by the active ribbon, which is now R.

Phase II: Let R' be the unique ribbon such that $\text{diag}(\text{head}(R')) = \text{diag}(\text{head}(R)) + k$ which is removable for the shape of $Y - H$. Since $\text{row}(s) > 1$, the letter m is not ejected in Remark 9. Theorem 11 implies that a ribbon of value m is not ejected by the Stanton-White reverse row insertion. But this is precisely the condition that R' exists. The horizontal ribbon strip H' (as a skew shape) is the union of R' and the shape $H - R$ which extends R'. The part of $H - R$ that is shown, has standardization given by the ribbons $R'_{\ell+1}$ through $R'_{\ell+c}$ from Phase I.

Example 14. Here is R' (indicated by x) extended by (part of) $\text{std}(H - R)$.

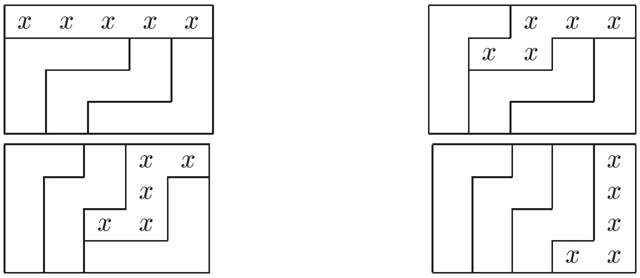


Phase III: The tiling of the horizontal ribbon strip H' given by the combination of R' and $\text{std}(H - R)$ computed above, is not equal to $\text{std}(H')$. To get the latter, one must switch the ribbon R' to the southeast past these same ribbons $R'_{\ell+1}, \dots, R'_{\ell+c}$ in $\text{std}(H - R)$. These are precisely the ribbons in $\text{std}(H - R)$ that share a diagonal with R' since $\text{diag}(\text{head}(R')) = \text{diag}(\text{head}(R)) + k$. Call J the ribbon in $\text{std}(H')$ that is the final resting place of R' under these switches. We have

$$\begin{aligned} \text{spin}(H') - \text{spin}(J) &= \text{spin}(H - R) + c \\ \text{spin}(J) &= \text{spin}(R') + c \end{aligned} \tag{4.5}$$

Putting together (4.4) and (4.5), (4.2) follows.

Example 15. $\mathrm{std}(H')$ is computed by switching the ribbon of x's, to the southeast.



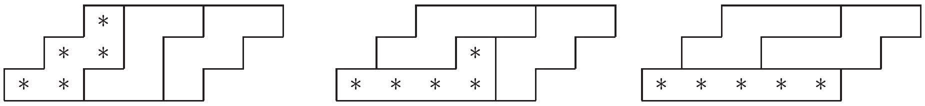
This is the local picture for the horizontal ribbon strip H' of value 3. J is the active ribbon in the last tableau.

The proof of (4.1) has been reduced to the case that $\mathrm{row}(s) = 1$. In the reverse row insertion in Remark 9 the letter m is ejected from $\mathrm{quot}_k(Y)$. By Theorem 11 this means that a ribbon of value m is ejected by the Stanton-White reverse row insertion. Thus Phase I proceeds as before, but during Phase II the ribbon R' does not exist, and the ribbon R_ℓ is ejected from the tableau. Thus $H' = H - R$. The rest of the tableau is unchanged; more precisely, Y and Y' agree except in the horizontal ribbon strips H and H'. From (4.4) one has

$$\begin{aligned}\mathrm{spin}(Y) - \mathrm{spin}(Y') - \mathrm{spin}(R) &= \mathrm{spin}(H) - \mathrm{spin}(H') - \mathrm{spin}(R) \\ &= 2c.\end{aligned}$$

In this case, by examining the k-quotient map one sees that $c = \mathrm{inv}(s, \mathrm{quot}_k(\mu))$ (where recall that $c + 1$ is the number of ribbons in $\mathrm{std}(H)$ that intersect R). This finishes the proof of (4.1).

Example 16. We shall recycle the above example. We will use the tableau Y_μ in Example 8 but with all 3's removed, and for which the active ribbon is indicated by $+$. We will calculate the change of spin in the horizontal ribbon strip H of value 2, since the value 2 is ejected in passing from $\mathrm{quot}_k(Y)$ to $\mathrm{quot}_k(Y')$. Below the 6-th, 7-th, and 8-th ribbons in H are depicted below.



The cells in $\mathrm{quot}_k(H)$ (where the 3's have been removed and H is the horizontal ribbon strip of 2's) are numbered in $\preceq$-increasing order in the following diagram:

The cell to be removed is labeled 6. It has inversions with cells labeled 7 and 8. This corresponds to the fact that the 7-th and 8-th ribbons in H were switched with the 6-th one during Phase I.

5. Comments

There should be some natural graded $GL(n)$-module whose character is $\widetilde{G}_\mu[X_n; q]$ for sufficiently large n and $X_n = (x_1, \dots, x_n)$, and some natural graded module for the appropriate symmetric group whose graded character is $W(\mu^\bullet; q)$. When $\mathrm{quot}_k(\mu)$ consists of single rows, the graded symmetric group module is given by the coordinate ring of the schematic intersection of the closure of a nilpotent conjugacy class in $gl(n)$, with the diagonal matrices [4].

When $\mu^\bullet = \mathrm{quot}_k(\mu)$ consists of rectangles, conjecturally such a graded $GL(n)$-module is given by the tensor product of finite-dimensional $U_q(\widehat{sl}_n)$-modules which under restriction to $U_q(sl_n)$ are irreducible integrable highest weight modules with highest weight corresponding to a rectangular partition [6]. These modules have a natural crystal basis given by so-called paths, which possess a statistic called the energy function. There should be some nice bijection from paths to semistandard ribbon tableaux that is weight-preserving and sends energy to cospin. In the case that $\mathrm{quot}_k(\mu) = ((1), (1), \dots, (1))$ and the weight is $(1, 1, \dots, 1) \in \mathbb{Z}^k$, the bijection that sends major index to the inversion number [2] gives such a path-to-ribbon tableau bijection.

References

[1] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, T. Welsh, Branching functions of $A^{(1)}_{n-1}$ and Jantzen-Seitz problem for Ariki-Koike algebras, Adv. Math. **141** (1999) 322–365. 2.2

[2] D. Foata and M.-P. Schützenberger, Major index and inversion number of permutations, Math. Nachr. **83** (1978), 143–159. 5

[3] S. Fomin and D. Stanton, Rim hook lattices, St. Petersburg Math. J. **9** (1998) 1007–1016. 2.4

[4] A. M. Garsia and C. Procesi, On certain graded S_n-modules and the q-Kostka polynomials, Adv. Math. **94** (1992) 82–138. 1, 3.3, 3.4, 5

[5] K. Killpatrick, Recursions for the q-Kostka Polynomials, Ph. D. Thesis, University of Minnesota, 1998. 3.3

[6] A. N. Kirillov, A. Schilling, and M. Shimozono, Various representations of the generalized Kostka polynomials, The Andrews Festschrift (Maratea, 1998), Sém. Lothar. Combin. **42** (1999), Art. B42j, 19 pp. 1, 5

[7] M. Kashiwara and T. Tanisaki, Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, math.RT/9908153. 1

[8] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties, J. Math. Phys. **38** (1997) 1041–1068. 1, 1, 2.5, 2.5, 3.3

[9] B. Leclerc and J.-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials, Adv. Studies in Pure Math. **28** (2000) 155–220. 1

[10] D. Stanton and D. E. White, A Schensted algorithm for rim hook tableaux, J. Combin. Th. Series A **40** (1985) 211–247. 4.3, 4.3

(A. Schilling) Building 2, room 279, Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139–4307

E-mail address: `anne@math.mit.edu`

URL: `http://www-math.mit.edu/~anne/`

(M. Shimozono) Department of Mathematics, Virginia Tech, Blacksburg, VA 24061–0123

E-mail address: `mshimo@math.vt.edu`

URL: `http://www.math.vt.edu/people/mshimo/`

(D. E. White) School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St SE, Minneapolis, MN 55455–0488

E-mail address: `white@math.umn.edu`

URL: `http://www.math.umn.edu/~white/`