A SIMPLE INTRODUCTION TO CRYSTALS $B^{2,s}$ FOR KIRILLOV-RESHETIKHIN MODULES OF TYPE $D_n^{(1)}$

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ABSTRACT. The Kirillov–Reshetikhin modules $W^{r,s}$ are finite-dimensional representations of quantum affine algebras $U_q'(\mathfrak{g})$, labeled by a Dynkin node r of the affine Kac–Moody algebra $\mathfrak{g}$ and a positive integer s. In this paper we explain the combinatorial structure of the crystal basis $B^{2,s}$ corresponding to $W^{2,s}$ for the algebra of type $D_n^{(1)}$.

Proofs of all claims, as well as more specific details of all constructions, may be found in [16].

1. INTRODUCTION

At the workshop on the Combinatorial Aspect of Integrable Systems held at RIMS Kyoto, one of the recurring themes was the $X = M$ conjecture of [1, 2]. Briefly, this conjecture states that the one-dimensional configuration sums X of a certain class of lattice models can be expressed as fermonic formulas M, reflecting the corner transfer matrix method and the Bethe ansatz as methods for solving these lattice models. The combinatorial tools of these methods are Young tableaux/crystal bases and rigged configurations, respectively. The following table summarizes the three regimes of this conjecture.

formulas	X : 1-D sum	M : fermionic formula
stat. mech. methods	CTM	Bethe ansatz
comb. objects	tableaux/crystals	rigged configurations

More specifically, the theory of crystal bases is used to label the highest weight vectors of irreducible representations (i.e., Bethe vectors) of a certain algebra by crystal basis elements. Since each Bethe vector corresponds to a solution of the Bethe equations and these solutions are indexed by rigged configurations, there should be a natural bijection between highest weight crystal elements and rigged configurations. Such bijections have been found by Kirillov and Reshetikhin [7] for type $A_n^{(1)}$ (see also [8]), and later for all nonexceptional types for the vector representation [10] and symmetric powers [15]. For type $D_n^{(1)}$ the bijection was given in [14] for the fundamental representations.

The $X = M$ conjecture depends upon the existence of the crystals $B^{r,s}$ for the Kirillov–Reshetikhin modules $W^{r,s}$. The Kirillov–Reshetikhin (KR) modules are finite-dimensional irreducible representations of quantum affine algebras $U_q'(\mathfrak{g})$. In general, it is not known yet whether the $B^{r,s}$ exist and what their combinatorial structure is. It is the purpose of this note to give the combinatorial structure of $B^{2,s}$ of type $D_n^{(1)}$. The KR crystals of type $A_n^{(1)}$ have been explicitly described [4, 13], as well as $B^{r,1}$ and $B^{1,s}$ for most types [4, 6]. Furthermore, according to the theory of virtual crystals [11, 12], the following algebra

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embeddings have been explicitly extended to the crystals of their KR modules:

$$\begin{array}{rcl} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} & \hookrightarrow & A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} & \hookrightarrow & D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} & \hookrightarrow & E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} & \hookrightarrow & D_4^{(1)}. \end{array}$$

The next case to explore is therefore $B^{2,s}$ for type $D_n^{(1)}$, which is the focus of this paper. Here, we present the combinatorial construction of $B^{2,s}$ assuming existence as recently given in [16]. The combinatorial crystal is denoted by $\tilde{B}^{2,s}$; we illustrate our main definition with examples. Proofs and further details can be found in [16]. The main result of [16] is:

Theorem 1.1. *If $B^{2,s}$ exists with the properties as in Conjecture 2.1, then $\tilde{B}^{2,s} \cong B^{2,s}$.*

2. Review

For background on quantum groups, crystal bases, perfect crystals, and other well-understood concepts, please refer to [16] or any of the standard references on these topics.

The fermionic formulas suggest not only the existence of the crystals $B^{r,s}$, but also several conjectures about the structure of these crystals as well [1]. In the case of $B^{2,s}$, this specializes to

Conjecture 2.1 ([1]). *The crystal $B^{2,s}$ of type $D_n^{(1)}$ exists and has the following properties:*

(1) *As a classical crystal $B^{2,s}$ decomposes as $B^{2,s} \cong \bigoplus_{k=0}^{s} B(k\Lambda_2)$.*
(2) *$B^{2,s}$ is perfect of level s.*
(3) *$B^{2,s}$ is equipped with an energy function $D_{B^{2,s}}$ such that $D_{B^{2,s}}(b) = k - s$ if b is in the component of $B(k\Lambda_2)$ (in accordance with the energy D as in* [16]*).*

To construct $\tilde{B}^{2,s}$ so that it satisfies these properties, we first find a way to label the vertices of the crystal. Our approach is to define a set of rules for what a legal "affine tableau" is, and then show that this set is in bijection with the direct sum $\bigoplus_{k=0}^{s} B(k\Lambda_2)$.

This bijection provides the action of the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \le i \le n$, but we still need to know the action of $\tilde{e}_0$ and $\tilde{f}_0$. To define these crystal operators, we use an auxilliary construction called the branching component graph. It can be shown that the resulting affine crystal $\tilde{B}^{2,s}$ is perfect of level s. In fact it was proved in [16] that this is the unique perfect level s crystal for which the energy function is as stated in Conjecture 2.1.

3. Affine tableaux

We briefly recall the labelling by tableaux of the vertices of classical highest weight crystals $B(k\Lambda_2)$ of highest weight $k\Lambda_2$, following the construction by Kashiwara and Nakashima [5]. Each crystal element can be represented by a tableau of shape $\lambda = (k,k)$ on the partially ordered alphabet

$$1 < 2 < \cdots < n-1 < \begin{matrix} n \\ \bar{n} \end{matrix} < \overline{n-1} < \cdots \bar{2} < \bar{1}$$

such that the following conditions hold [3, page 202]:

Criterion 3.1.

(1) *If ab is in the filling, then $a \le b$;*
(2) *If $\begin{smallmatrix} a \\ b \end{smallmatrix}$ is in the filling, then $b \nleq a$;*

(3) *No configuration of the form* $\begin{smallmatrix} a & a \\ & \bar{a} \end{smallmatrix}$ *or* $\begin{smallmatrix} a & \\ \bar{a} & \bar{a} \end{smallmatrix}$ *appears;*
(4) *No configuration of the form* $\begin{smallmatrix} n-1 & n \\ n & n-1 \end{smallmatrix}$ *or* $\begin{smallmatrix} n-1 & \bar{n} \\ \bar{n} & n-1 \end{smallmatrix}$ *appears;*
(5) *No configuration of the form* $\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}$ *appears.*

Note that for $k \geq 2$, condition 5 follows from conditions 1 and 3.

We define the set of affine tableau in $\tilde{B}^{2,s}$ by removing parts 3 and 5 from Criterion 3.1. The bijection between $\tilde{B}^{2,s}$ and $\bigoplus_{k=0}^{s} B(k\Lambda_2)$ is as follows. Given an affine tableau T which is not a classical tableau (i.e., a tableau that satisfies parts 1, 2, and 4 of 3.1, but violates part 3 or 5) there must be a configuration of the form $\begin{smallmatrix} a & a \\ & \bar{a} \end{smallmatrix}$, $\begin{smallmatrix} a & \\ \bar{a} & \bar{a} \end{smallmatrix}$ or $\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}$. Remove columns of the form $\begin{smallmatrix} a \\ \bar{a} \end{smallmatrix}$ (possibly with $a = 1$) until the resulting tableau satisfies Criterion 3.1. It can be shown that this procedure gives a well-defined bijection between the two sets.

The following examples are taken from $\tilde{B}^{2,5}$ for $D_4^{(1)}$.

Example 3.2. The affine tableau $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}$ corresponds to the classical tableau $\begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}$, by removing the second and third columns.

It is easy to see that for any affine tableau the removed columns must be adjacent, as they are in these examples.

Example 3.3. The affine tableau $\begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 3 & 3 & \bar{4} \\ \hline 4 & \bar{3} & \bar{3} & \bar{2} & \bar{1} \\ \hline \end{array}$ corresponds to the classical tableau $\begin{array}{|c|c|c|c|}\hline 2 & 3 & 3 & \bar{4} \\ \hline 4 & \bar{3} & \bar{2} & \bar{1} \\ \hline \end{array}$ by removing either the second or the third column.

As the above example indicates, if there is a choice about which column to remove, it has no effect on the outcome.

Example 3.4. The classical tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{4} & \bar{2} \\ \hline \end{array}$ corresponds to the affine tableau $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 2 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\ \hline \end{array}$.

While we could choose to add columns of the form $\begin{smallmatrix} 2 \\ \bar{2} \end{smallmatrix}$ either to the middle or to the right side of the first tableau, either choice results in the same affine tableau.

Example 3.5. The classical tableau $\begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}$ corresponds to the affine tableau $\begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}$.

By part 1 of Criterion 3.1, the only place that a column of the form $\begin{smallmatrix} a \\ \bar{a} \end{smallmatrix}$ may be inserted is between the first and second columns of t. However, we may choose between using this to create a configuration of either of the forms $\begin{smallmatrix} a & a \\ & \bar{a} \end{smallmatrix}$ or $\begin{smallmatrix} a & \\ \bar{a} & \bar{a} \end{smallmatrix}$. Once again, this "choice" does not affect the outcome.

4. THE BRANCHING COMPONENT GRAPH

Since the Dynkin diagram for type $D_n^{(1)}$ has a graph automorphism interchanging nodes 0 and 1, we know that interchanging the role of 1-arrows and 0-arrows in $\tilde{B}^{2,s}$ will produce an affine crystal isomorphic to $\tilde{B}^{2,s}$. We may use this fact to our advantage at a

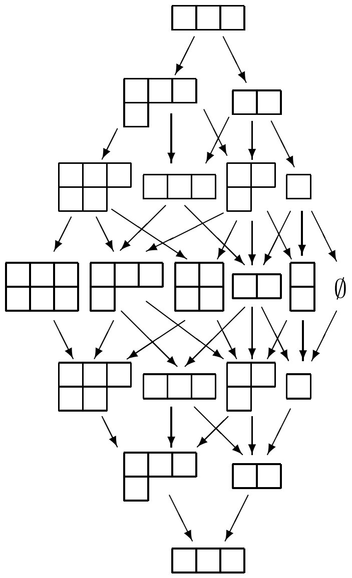

FIGURE 1. Branching component graph $\mathcal{BC}(3\Lambda_2)$

larger scale by considering the D_{n-1}-crystals that result from removing the 1-arrows from $\bigoplus_{k=0}^{s} B(k\Lambda_2)$, since this direct sum is isomorphic to $\tilde{B}^{2,s}$ with the 0-arrows removed.

The branching component graph of $\tilde{B}^{2,s}$, denoted $\mathcal{BC}(\tilde{B}^{2,s})$, is defined as follows. Its vertices correspond to the D_{n-1}-crystals that remain connected after removing all 0-arrows and 1-arrows from $\tilde{B}^{2,s}$; we label the vertices (non-uniquely) by the partition λ indicating the classical highest weight of the corresponding $U_q(D_{n-1})$-crystal. The edges of $\mathcal{BC}(\tilde{B}^{2,s})$ are defined by placing an edge from v to w if there is a tableau $b \in B(v)$ such that $\tilde{f}_1(b) \in B(w)$, where $B(v)$ denotes the set of tableaux contained in the D_{n-1}-crystal indexed by v.

It suffices to describe the effect of removing the 1-arrows from $B(k\Lambda_2)$ for arbitrary k. We denote this branching component graph by $\mathcal{BC}(k\Lambda_2)$, and use v_k to denote the "highest weight branching vertex", i.e., the branching vertex such that the highest weight tableaux $b_{k\Lambda_2} \in B(v_k)$.

An intuitive way to construct $\mathcal{BC}(k\Lambda_2)$ is as follows. Begin with a $1 \times k$ rectangle, which labels v_k. For $1 \leq j \leq k$, the partitions labeling the vertices of rank j are those which are contained in a $2 \times k$ rectangle and which are joined by an edge in Young's lattice to some partition labeling a vertex in rank $j-1$. In each rank, the partitions appear with multiplicity one. For $k+1 \leq j \leq 2k$, the partitions in rank j are the same as those in rank $2k-j$, again with multiplicity one. Finally, there is an edge from a vertex v of rank j to a vertex w of rank $j+1$ precisely when the corresponding partitions are joined by an edge in Young's lattice.

Example 4.1. Figure 1 depicts $\mathcal{BC}(3\Lambda_2)$.

There is a unique inclusion of $\mathcal{BC}(k\Lambda_2)$ in $\mathcal{BC}((k+1)\Lambda_2)$ that agrees with the labelling of the vertices. We may define a rank function on all of $\mathcal{BC}(\tilde{B}^{2,s})$ by setting the rank of

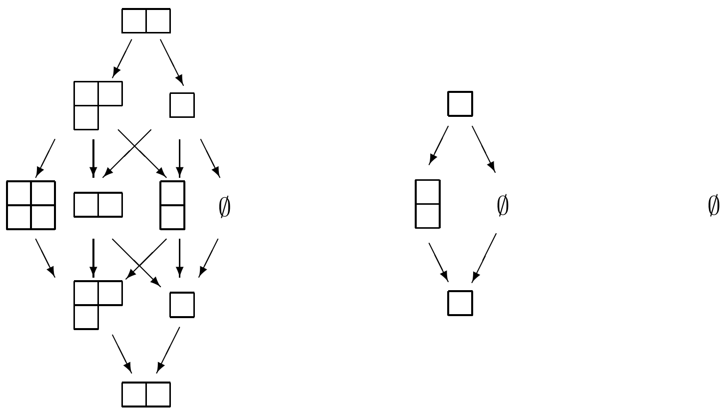

FIGURE 2. Branching component graph $\mathcal{BC}(\tilde{B}^{2,2})$

a vertex to the rank of its image in $\mathcal{BC}(s\Lambda_2)$ under the appropriate composition of these inclusions. For example, every vertex labelled by $\emptyset$ always has rank s in $\mathcal{BC}(\tilde{B}^{2,s})$.

Example 4.2. Figure 2 depicts $\mathcal{BC}(\tilde{B}^{2,2})$, which is the union of $\mathcal{BC}(0)$, $\mathcal{BC}(\Lambda_2)$, and $\mathcal{BC}(2\Lambda_2)$.

5. Affine Kashiwara operators

In this section we describe how to "overlay" a set of arrows, called F_0 arrows, on $\mathcal{BC}(\tilde{B}^{2,s})$ in a way that specifies $\tilde{e}_0$ and $\tilde{f}_0$. Let $v \in \mathcal{BC}(\tilde{B}^{2,s})$ be a vertex of global rank j in $\mathcal{BC}(k\Lambda_2)$ associated with the partition (λ_1, λ_2). Place an F_0 arrow from v to the following vertices, *if they exist*:

- the vertex of global rank $j-1$ in $\mathcal{BC}((k-1)\Lambda_2)$ with shape $(\lambda_1 - 1, \lambda_2)$;
- the vertex of global rank $j-1$ in $\mathcal{BC}(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 - 1)$;
- the vertex of global rank $j-1$ in $\mathcal{BC}((k+1)\Lambda_2)$ with shape $(\lambda_1 + 1, \lambda_2)$;
- the vertex of global rank $j-1$ in $\mathcal{BC}(k\Lambda_2)$ with shape $(\lambda_1, \lambda_2 + 1)$.

The directed graph that consists of the vertices of $\mathcal{BC}(\tilde{B}^{2,s})$ and the F_0 arrows is isomorphic to $\mathcal{BC}(\tilde{B}^{2,s})$. Via this graph isomorphism, which we denote σ, we may define $\tilde{f}_0$ for $\tilde{B}^{2,s}$. Let $b \in B(v)$ be a tableau in $\tilde{B}^{2,s}$. Note that $B(v)$ is isomorphic to $B(\sigma(v))$ as a D_{n-1}-crystal; let $b' \in B(\sigma(v))$ denote the tableau corresponding to b under this isomorphism. We may have $\tilde{f}_1(b') = c' \in B(w)$ for some branching vertex w, or we may have $\tilde{f}_1(b') = 0$. In the former case, we say that $\tilde{f}_0(b) = c$, where c corresponds to c' under the isomorphism between $B(w)$ and $B(\sigma(w))$; in the latter case, $\tilde{f}_0(b) = 0$. By the definition of crystals, this also determines $\tilde{e}_0$.

Example 5.1. In Figure 3 we have $\mathcal{BC}(\tilde{B}^{2,2})$ with the original arrows removed and the F_0 arrows superimposed.

Of course, we could have chosen to define the graph isomorphism in terms of the branching vertices, and let the definition of the F_0 arrows follow. In fact, we did exactly that in [16], where σ is used to denote the automorphism of the vertices of $\tilde{B}^{2,s}$ corresponding to interchanging nodes 0 and 1 of the Dynkin diagram.

We now present some examples taken from $\tilde{B}^{2,2}$.

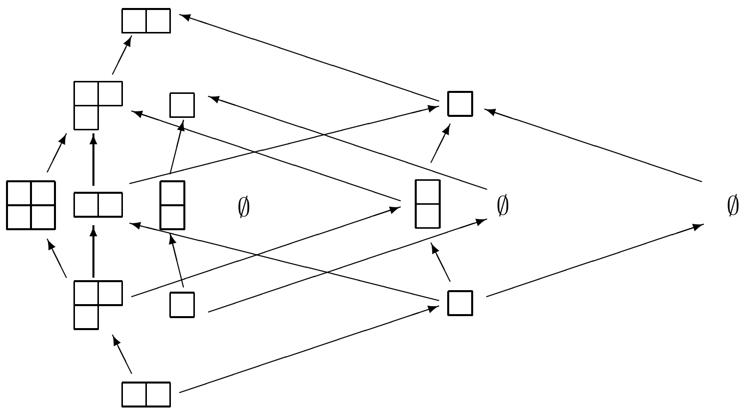

FIGURE 3. $\mathcal{BC}(\tilde{B}^{2,2})$ with F_0 arrows

Example 5.2. Let $b = \begin{smallmatrix} 1 & 2 \\ \bar{1} & \bar{1} \end{smallmatrix}$, so $b \in B(v)$ where v is the branching vertex of shape $(1,0)$ with global rank 3 in $\mathcal{BC}(\Lambda_2)$. We see from Figures 2 and 3 that $\sigma(v)$ is the vertex with the same shape with rank 1 in $\mathcal{BC}(2\Lambda_2)$. The corresponding tableau in $\sigma(v)$ is $b' = \begin{smallmatrix} 1 & 2 \\ 2 & \bar{2} \end{smallmatrix}$, and $c' = \tilde{f}_1(b') = \begin{smallmatrix} 1 & 2 \\ 2 & \bar{1} \end{smallmatrix}$. The branching vertex containing c' is the vertex of shape $(1,1)$ with rank 2 in $\mathcal{BC}(2\Lambda_2)$, which is fixed under σ, so $c = c'$. Therefore, $\tilde{f}_0(b) = \begin{smallmatrix} 1 & 2 \\ 2 & \bar{1} \end{smallmatrix}$.

Example 5.3. Let $b = \begin{smallmatrix} 3 & 3 \\ \bar{1} & \bar{1} \end{smallmatrix}$, so $b \in B(v)$ where v is the branching vertex of shape $(2,0)$ with rank 4 in $\mathcal{BC}(2\Lambda_2)$. We see from Figures 2 and 3 that $\sigma(v)$ is the vertex of the same shape with rank 0 in $\mathcal{BC}(2\Lambda_2)$. The corresponding tableau in $\sigma(v)$ is $b' = \begin{smallmatrix} 1 & 1 \\ 3 & 3 \end{smallmatrix}$, and $c' = \tilde{f}_1(b') = \begin{smallmatrix} 1 & 2 \\ 3 & 3 \end{smallmatrix}$. The branching vertex containing c' is the vertex of shape $(2,1)$ with rank 1 in $\mathcal{BC}(2\Lambda_2)$. Its image under σ is the vertex of the same shape with rank 3 in $\mathcal{BC}(2\Lambda_2)$, so $\tilde{f}_0(b) = c = \begin{smallmatrix} 2 & 3 \\ 3 & \bar{1} \end{smallmatrix}$.

Example 5.4. Let $b_{k\Lambda_2}$ denote the classical highest weight tableau of $B(k\Lambda_2) \subset \tilde{B}^{2,s}$. Then $\tilde{f}_0(b_{k\Lambda_2}) = b_{(k+1)\Lambda_2}$ for $0 \leq k \leq s-1$.

6. PERFECTNESS

Several conditions must be satisfied for a crystal B to be a perfect crystal of level ℓ, but the most significant challenge is in the condition that the maps ε and φ from $B_{\min}$ to $(P_{\mathrm{cl}}^+)_\ell$ are bijective. We briefly recall the definition of these sets and maps below; for more detail see [16] or [4].

For a crystal basis element $b \in B$, define the weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i,$$

where

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq \emptyset\}$$
$$\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq \emptyset\}.$$

The level of a weight Λ is $\langle c, \Lambda \rangle$, where $c = h_0 + h_1 + h_{n-1} + h_n + \sum_{n=2}^{n-2} 2h_i$ is the canonical central element of the algebra of type $D_n^{(1)}$. The set of minimal vertices, denoted

$B_{\min}$, is the set of crystal elements b for which $\langle c, \varepsilon(b)\rangle$ is minimal. Finally, define $(P_{\mathrm{cl}}^+)_\ell$ to be the set of level ℓ weights Λ with no δ component for which $\langle h_i, \Lambda\rangle \geq 0$ for all $i \in I$.

We now outline the construction of a $2 \times s$ tableau T such that given any level s weight Λ, we have $\varepsilon(T) = \varphi(T) = \Lambda$. It was shown in [16] that these are precisely the tableaux in $B_{\min}$.

For $i = 0, \dots, n$, let $k_i = \langle h_i, \lambda\rangle$. We first construct a tableau $T_{\lambda'}$ corresponding to the weight $\lambda' = \sum_{i=2}^n k_i \Lambda_i$. We begin with the middle $k_{n-1} + k_n$ columns of $T_{\lambda'}$. If $k_{n-1} + k_n$ is even and $k_n \geq k_{n-1}$, these columns of $T_{\lambda'}$ are

$$\underbrace{\begin{array}{ccc} n-2 & \cdots & n-2 \\ n-1 & & n-1 \end{array}}_{k_{n-1}} \underbrace{\begin{array}{ccc} n-1 & \cdots & n-1 \\ n & & n \end{array}}_{(k_n-k_{n-1})/2} \underbrace{\begin{array}{ccc} \overline{n} & \cdots & \overline{n} \\ \overline{n-1} & & \overline{n-1} \end{array}}_{(k_n-k_{n-1})/2} \underbrace{\begin{array}{ccc} \overline{n-1} & \cdots & \overline{n-1} \\ \overline{n-2} & & \overline{n-2} \end{array}}_{k_{n-1}}$$

If $k_{n-1} + k_n$ is odd and $k_n \geq k_{n-1}$, we have

$$\underbrace{\begin{array}{ccc} n-2 & \cdots & n-2 \\ n-1 & & n-1 \end{array}}_{k_{n-1}} \underbrace{\begin{array}{ccc} n-1 & \cdots & n-1 \\ n & & n \end{array}}_{(k_n-k_{n-1}-1)/2} \begin{array}{c} \bar{n} \\ n \end{array} \underbrace{\begin{array}{ccc} \overline{n} & \cdots & \overline{n} \\ \overline{n-1} & & \overline{n-1} \end{array}}_{(k_n-k_{n-1}-1)/2} \underbrace{\begin{array}{ccc} \overline{n-1} & \cdots & \overline{n-1} \\ \overline{n-2} & & \overline{n-2} \end{array}}_{k_{n-1}}$$

In either case, if $k_n < k_{n-1}$, interchange n and $\bar{n}$, and k_n and k_{n-1} in the above configurations.

Next we put a configuration of the form

$$\underbrace{\begin{array}{ccc} 1 & \cdots & 1 \\ 2 & & 2 \end{array}}_{k_2} \underbrace{\begin{array}{ccc} 2 & \cdots & 2 \\ 3 & & 3 \end{array}}_{k_3} \quad \cdots \quad \underbrace{\begin{array}{ccc} n-3 & \cdots & n-3 \\ n-2 & & n-2 \end{array}}_{k_{n-2}}$$

on the left, and a configuration of the form

$$\underbrace{\begin{array}{ccc} \overline{n-2} & \cdots & \overline{n-2} \\ \overline{n-3} & & \overline{n-3} \end{array}}_{k_{n-2}} \underbrace{\begin{array}{ccc} \overline{n-3} & \cdots & \overline{n-3} \\ \overline{n-4} & & \overline{n-4} \end{array}}_{k_{n-3}} \quad \cdots \quad \underbrace{\begin{array}{ccc} \bar{2} & \cdots & \bar{2} \\ \bar{1} & & \bar{1} \end{array}}_{k_2}$$

on the right.

We now use Lecouvey D equivalence as in [9] or type D sliding as in [16] to change this tableau into a skew tableau of shape $(s - k_0, s - k_0 - k_1)/(k_1)$. If $k_1 > s - k_0 - k_1$ (i.e., $k_1 - (s - k_0 - k_1) = 2k_1 + k_0 - s > 0$), place a configuration of the following form in the empty spaces in the middle of this skew tableau:

$$\underbrace{\begin{array}{cccccc} 1 & \cdots & 1 & 2 & \cdots & 2 \\ \bar{2} & & \bar{2} & \bar{1} & & \bar{1} \end{array}}_{2k_1+k_0-s} \qquad \text{if } 2k_1 + k_0 - s \text{ is even,}$$

$$\underbrace{\begin{array}{ccccccc} 1 & \cdots & 1 & 2 & 2 & \cdots & 2 \\ \bar{2} & & \bar{2} & \bar{2} & \bar{1} & & \bar{1} \end{array}}_{2k_1+k_0-s} \qquad \text{if } 2k_1 + k_0 - s \text{ is odd,}$$

where the number of 1's equals the number of $\bar{1}$'s and the number of 2's equals the number of $\bar{2}$'s.

If $s - k_0$ is odd, the middle column of the tableau constructed so far is $\begin{smallmatrix} a \\ \bar{a} \end{smallmatrix}$ for $1 \leq a \leq n$ or $\begin{smallmatrix} \bar{n} \\ n \end{smallmatrix}$. Whatever it is, simply insert k_0 of this column into the tableau next to the middle column (cf. Section 3). If $s - k_0$ is even, the middle two columns are of the form $\begin{smallmatrix} a & \bar{b} \\ b & \bar{a} \end{smallmatrix}$ for some letters a and b (it is possible that b is barred, in which case $\bar{b}$ is the corresponding unbarred letter). In this case, simply add k_0 columns of the form $\begin{smallmatrix} a \\ \bar{a} \end{smallmatrix}$ between these columns.

We provide a few examples with details of the construction of the tableaux, followed by examples with less detail in Table 1.

Example 6.1. Let $n = 5$, and consider the weight $\Lambda_0 + 2\Lambda_1 + \Lambda_2 + 2\Lambda_4 + \Lambda_5$. This weight has level $1 + 2 + 2 \cdot 1 + 2 + 1 = 8$, so our proceedure will result in a 2×8 tableau; i.e., a minimal tableau in $\tilde{B}^{2,8}$. Since $k_4 + k_5$ is odd and $k_5 < k_4$, we begin with $\begin{array}{|c|c|c|}\hline 3 & 5 & \bar{4} \\ \hline 4 & \bar{5} & \bar{3} \\ \hline\end{array}$. To incorporate Λ_2, we amend this tableau to get $\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & \bar{4} & \bar{2} \\ \hline 2 & 4 & \bar{5} & \bar{3} & \bar{1} \\ \hline\end{array}$. Applying the type D sliding algorithm twice and inserting 1's and $\bar{1}$'s gives us $\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 4 & 5 & \bar{4} & \bar{2} \\ \hline 2 & 4 & \bar{5} & \bar{4} & \bar{1} & \bar{1} & \bar{1} \\ \hline\end{array}$. Finally, we insert one column in the middle, which yields $\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 4 & 4 & 5 & \bar{4} & \bar{2} \\ \hline 2 & 4 & \bar{5} & \bar{4} & \bar{4} & \bar{1} & \bar{1} & \bar{1} \\ \hline\end{array}$.

Example 6.2. Let $n = 6$, and consider the weight $\Lambda_0 + 2\Lambda_1 + \Lambda_3 + 2\Lambda_6$. This weight has level $1 + 2 + 2 \cdot 1 + 2 = 7$, so we will have a 2×7 tableau at the end; i.e., a minimal tableau in $\tilde{B}^{2,7}$. We begin with the tableau corresponding to $2\Lambda_6$, which is $\begin{array}{|c|c|}\hline 5 & \bar{6} \\ \hline 6 & \bar{5} \\ \hline\end{array}$, and expand it thus an account of Λ_3: $\begin{array}{|c|c|c|c|}\hline 2 & 5 & \bar{6} & \bar{3} \\ \hline 3 & 6 & \bar{5} & \bar{2} \\ \hline\end{array}$. Type D sliding turns it into $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 6 & \bar{6} & \bar{3} \\ \hline 3 & 6 & \bar{6} & \bar{2} & \bar{1} & \bar{1} \\ \hline\end{array}$, and inserting one column gives us $\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 6 & \bar{6} & \bar{3} \\ \hline 3 & 6 & \bar{6} & \bar{2} & \bar{2} & \bar{1} & \bar{1} \\ \hline\end{array}$.

Example 6.3. Table 1 shows several weights and the corresponding tableaux. The first 11 entries are all the level 2 weights for $n = 4$.

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n	weight	level of weight	tableau
4	$2\Lambda_0$	2	$\begin{matrix} 1 & 1 \\ \bar{1} & \bar{1} \end{matrix}$
4	$2\Lambda_1$	2	$\begin{matrix} 1 & 2 \\ \bar{2} & \bar{1} \end{matrix}$
4	$2\Lambda_3$	2	$\begin{matrix} 3 & 4 \\ \bar{4} & \bar{3} \end{matrix}$
4	$2\Lambda_4$	2	$\begin{matrix} 3 & \bar{4} \\ 4 & \bar{3} \end{matrix}$
4	$\Lambda_0 + \Lambda_1$	2	$\begin{matrix} 2 & 2 \\ \bar{2} & \bar{2} \end{matrix}$
4	$\Lambda_0 + \Lambda_3$	2	$\begin{matrix} 4 & 4 \\ \bar{4} & \bar{4} \end{matrix}$
4	$\Lambda_0 + \Lambda_4$	2	$\begin{matrix} \bar{4} & \bar{4} \\ 4 & 4 \end{matrix}$
4	$\Lambda_1 + \Lambda_3$	2	$\begin{matrix} 1 & 4 \\ \bar{4} & \bar{1} \end{matrix}$
4	$\Lambda_1 + \Lambda_4$	2	$\begin{matrix} 1 & \bar{4} \\ 4 & \bar{1} \end{matrix}$
4	$\Lambda_3 + \Lambda_4$	2	$\begin{matrix} 2 & \bar{3} \\ 3 & \bar{2} \end{matrix}$
4	Λ_2	2	$\begin{matrix} 1 & \bar{2} \\ 2 & \bar{1} \end{matrix}$
7	$2\Lambda_3 + 3\Lambda_4$	10	$\begin{matrix} 2 & 2 & 3 & 3 & 3 & \bar{4} & \bar{4} & \bar{4} & \bar{3} & \bar{3} \\ 3 & 3 & 4 & 4 & 4 & \bar{3} & \bar{3} & \bar{3} & \bar{2} & \bar{2} \end{matrix}$
7	$2\Lambda_0 + \Lambda_2 + 2\Lambda_4 + \Lambda_5$	10	$\begin{matrix} 1 & 3 & 3 & 4 & 4 & 4 & \bar{5} & \bar{4} & \bar{4} & \bar{2} \\ 2 & 4 & 4 & 5 & \bar{4} & \bar{4} & \bar{4} & \bar{3} & \bar{3} & \bar{1} \end{matrix}$
7	$\Lambda_4 + 2\Lambda_6 + 7\Lambda_7$	11	$\begin{matrix} 3 & 5 & 5 & 6 & 6 & \bar{7} & \bar{7} & \bar{7} & \bar{6} & \bar{6} & \bar{4} \\ 4 & 6 & 6 & 7 & 7 & 7 & \bar{6} & \bar{6} & \bar{5} & \bar{5} & \bar{3} \end{matrix}$

TABLE 1. Weights and minimal tableaux

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