New Expressions for Level-restricted Kostka Polynomials

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Abstract. We show that the bijection between tableaux and rigged configurations introduced by Kirillov and Reshetikhin is well-behaved with respect to level-restriction. In particular we give an explicit characterization of the image of level-restricted tableaux under this bijection which yields new expressions for level-restricted Kostka polynomials. We conjecture that the bijection is also well-behaved with respect to skew shapes and give an explicit characterization of the image under the bijection. Type $\widehat{A}$ branching functions can be expressed in terms of limits of level-restricted Kostka polynomials, thereby also yielding new expressions for these branching functions.

1 Introduction

The Littlewood–Richardson coefficients are tensor product multiplicities defined as

$$c_R^\lambda = \dim \mathrm{Hom}_{sl_n}(V^\lambda : V^{R_1} \otimes \cdots \otimes V^{R_L}),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, $R = (R_1, \ldots, R_L)$ is a sequence of partitions, and, for ν a partition, V^ν is the irreducible integrable highest weight module of highest weight ν over sl_n.

There are two natural generalizations of the Littlewood–Richardson coefficients. There exist polynomials $c_R^\lambda(q)$ which specialize to c_R^λ when $q = 1$. For example in ref. [21] such q-analogues were introduced as multiplicities for graded $GL(n)$-modules supported in the closure of a nilpotent conjugacy class. Lascoux, Leclerc and Thibon [16] defined a q-analogue of c_R^λ as the generating function of ribbon tableaux with spin statistic. In ref. [17] these polynomials were shown to be parabolic Kazhdan–Lusztig polynomials for affine symmetric groups. Conjecturally [12], the q-analogues of refs. [21] and [16] coincide when R is a sequence of rectangles. When R is a sequence of rectangles the polynomials of [21] give rise to generalized Kostka polynomials. Generalized Kostka polynomials can also be defined as generating functions of Littlewood–Richardson tableaux with a charge statistic [19, 20], generalizing the charge of Lascoux and Schützenberger [15]. This is the definition used here.

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There also exist level-restricted analogues of the Littlewood–Richardson coefficients c_R^λ (see for example [4, 8, 10]). In the setting of Kashiwara's [9] crystal base theory level-restricted Littlewood–Richardson coefficients occur naturally by extending the $U_q(sl_n)$-crystal structure to a $U_q(\widehat{sl}_n')$-crystal structure [10]; here $\widehat{sl}_n'$ is the derived algebra of the affine Kac–Moody algebra $\widehat{sl}_n$. In ref. [5] the level-restricted Littlewood–Richardson coefficients are defined as multiplicities of an induction product of modules over Hecke algebras at a root of unity. Level-restriction is also a very important concept in the context of restricted-solid-on-solid models in statistical mechanics [2] and fusion models in conformal field theory [22].

In this note we consider the two generalizations together giving rise to level-restricted generalized Kostka polynomials $K^\ell_{\lambda R}(q)$, where ℓ is the level (in fact, we will only consider in detail level-restricted Kostka polynomials which correspond to all R_i being single rows; but the general case of R_i being rectangular is very similar, see [18]). In refs. [3, 6, 7, 14, 19, 23] fermionic formulas for $K^\ell_{\lambda R}(q)$ for special λ (and R) were conjectured. Fermionic formulas originate from the Bethe Ansatz and usually involve (relatively small) sums of products of q-binomial coefficients. Here we present fermionic formulas for all λ (proven in ref. [18]). Their structure is considerably more complex than the conjectured special cases.

Type $\widehat{A}$ branching functions can be expressed as limits of level-restricted Kostka polynomials. It is possible to take these limits explicitly in the fermionic expressions for the level-restricted Kostka polynomials, thereby yielding new expressions for these branching functions.

Our technique for finding and proving new formulas for the level-restricted (generalized) Kostka polynomials proceeds via a bijection between tableaux and rigged configurations first introduced in ref. [11]. In ref. [13] it was shown that this bijection is well-defined and there it was used to prove a fermionic formula for the (generalized) Kostka polynomials. In order to get a handle on the level-restricted (generalized) Kostka polynomials one needs to characterize the image of level-restricted tableaux under this bijection. This is done in Section 3 and yields immediately the desired new expressions for the level-restricted (generalized) Kostka polynomials given in Section 4. In Section 5 we mention that one can obtain new expressions for type $\widehat{A}$ branching functions from those of the level-restricted Kostka polynomials, and we present a conjecture for the explicit characterization of rigged configurations corresponding to "skew" tableaux.

For simplicity of presentation we restrict ourselves to the case that all R_i are single rows. The general rectangle case as well as all the details omitted here can be found in ref. [18] on which this extended abstract is based.

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2 Level-restricted Tableaux

Lascoux and Schützenberger [15] showed that the Kostka polynomials $K_{\lambda\mu}(q)$, labeled by a partition λ and an array of nonnegative integers μ, can be expressed as the generating function of column-strict tableaux with charge statistics

$$K_{\lambda\mu}(q) = \sum_{T\in \mathrm{CST}(\lambda;\mu)} q^{c(T)}.$$

Here $\mathrm{CST}(\lambda;\mu)$ is the set of all column-strict tableaux of shape λ and content μ, and $c(T)$ is the charge of tableau T.

From now on let us always assume that λ has n parts, some of which may be zero. Let ℓ be a positive integer. Level-ℓ restricted tableaux are defined as follows [5]. Let $T \in \mathrm{CST}(\lambda;\mu)$. The tableau T may be viewed as a sequence of shapes $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(L)} = \lambda$ where $\lambda^{(j)}$ is the shape of the subtableau of T based on the subalphabet $\{1,2,\ldots,j\}$. Then T is restricted of level ℓ if

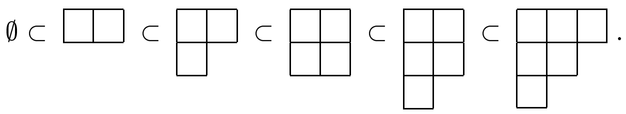
$$\lambda_1^{(j)} - \lambda_n^{(j-1)} \le \ell \qquad \text{for all } 1\le j\le L. \tag{1}$$

Example 1.

1. The sequence of shapes for the tableau

$$T = \begin{array}{lll} 1 & 1 & 5 \\ 2 & 3 & \\ 4 & & \end{array} \tag{2}$$

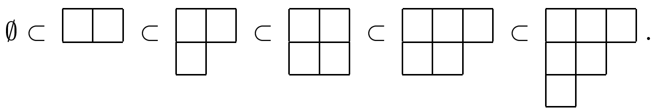
   is given by



   Hence, for $n = 3$, T is restricted of level 2.
   On the other hand, the tableau

$$\begin{array}{lll} 1 & 1 & 4 \\ 2 & 3 & \\ 5 & & \end{array}$$

   is not level-2 restricted since its sequence of shapes is given by



2. In the case of standard tableaux, that is, when $\mu = (1^{|\lambda|})$ so that each letter occurs only once, write $\mathrm{ST}(\lambda) = \mathrm{CST}(\lambda;\mu)$. For the set of level-$\ell$ restricted standard tableaux $\mathrm{ST}^{\ell}(\lambda)$ the above conditions simplify slightly.

Let $S \in \mathrm{ST}(\lambda)$ and associate the chain of shapes $\lambda^{(j)}$ as above. Since passing from $\lambda^{(j-1)}$ to $\lambda^{(j)}$ adds only one additional cell, the condition (1) simplifies to

$$\lambda_1^{(j)} - \lambda_n^{(j)} \leq \ell \qquad \text{for all } 1 \leq j \leq L.$$

The level-ℓ restricted Kostka polynomials are defined as the generating functions of level-ℓ restricted tableaux with charge statistics

$$K^{\ell}_{\lambda\mu}(q) = \sum_{T \in \mathrm{CST}^{\ell}(\lambda;\mu)} q^{c(T)}. \tag{3}$$

3 Level-restricted Rigged Configurations

In 1988 Kirillov and Reshetikhin [11] introduced a bijection between column-strict tableaux and rigged configurations. Recently it was shown that this bijection is well-defined and preserves the statistics [13]. In particular this led to a proof of a fermionic formula for the (generalized) Kostka polynomials. Here we show that the bijection is well-behaved with respect to level-restriction which yields new expressions for the level-restricted (generalized) Kostka polynomials.

3.1 Rigged Configurations

Let us review the definition of rigged configurations.

A $(\lambda;\mu)$-configuration is a sequence of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \ldots)$ with the size constraints

$$|\nu^{(k)}| = \sum_{j>k} \lambda_j$$

for $k \geq 0$ where by convention $\nu^{(0)} = \mu$. If λ has at most n parts all partitions $\nu^{(k)}$ for $k \geq n$ are empty. For a partition ρ, define $m_i(\rho)$ to be the number of parts equal to i and

$$Q_i(\rho) = \rho_1^t + \rho_2^t + \ldots + \rho_i^t = \sum_{j \geq 1} \min\{i, \rho_j\},$$

the size of the first i columns of ρ. The definition of $Q_i(\rho)$ also extends to arrays of numbers ρ using the second expression. The vacancy numbers for the $(\lambda;\mu)$-configuration ν are the numbers (indexed by $k \geq 1$ and $i \geq 0$) defined by

$$P_i^{(k)}(\nu) = Q_i(\nu^{(k-1)}) - 2Q_i(\nu^{(k)}) + Q_i(\nu^{(k+1)}). \tag{4}$$

In particular $P_0^{(k)}(\nu) = 0$ for all $k \geq 1$. The $(\lambda;\mu)$-configuration ν is said to be admissible if $P_i^{(k)}(\nu) \geq 0$ for all $k, i \geq 1$, and the set of admissible $(\lambda;\mu)$-configurations is denoted by $\mathrm{C}(\lambda;\mu)$. Following [12, (3.2)], set

$$cc(\nu) = \sum_{k,i \geq 1} \alpha_i^{(k)} (\alpha_i^{(k)} - \alpha_i^{(k+1)})$$

where $\alpha_i^{(k)}$ is the size of the i-th column in $\nu^{(k)}$. Define the charge $c(\nu)$ of a configuration $\nu \in \mathrm{C}(\lambda;\mu)$ by

$$c(\nu) = ||\mu|| - cc(\nu) - |P|$$

$$\text{with} \quad ||\mu|| = \sum_{1 \le i < j \le L} \min\{\mu_i, \mu_j\} \quad \text{and} \quad |P| = \sum_{k,i \ge 1} m_i(\nu) P_i^{(k)}(\nu).$$

Observe that $c(\nu)$ depends on both ν and μ but $cc(\nu)$ depends only on ν.

Rigged configurations can be defined as certain labelings of the rows of the partitions in a configuration. For this purpose one should view a partition as a multiset of positive integers. A rigged partition is by definition a finite multiset of pairs (i, x) where i is a positive integer and x is a nonnegative integer. The pairs (i, x) are referred to as strings; i is referred to as the length of the string and x as the label or quantum number of the string. A rigged partition is said to be a rigging of the partition ρ if the multiset consisting of the lengths of the strings, is the partition ρ. So a rigging of ρ is a labeling of the parts of ρ by nonnegative integers, where one identifies labelings that differ only by permuting labels among equal-sized parts of ρ.

A rigging J of the $(\lambda;\mu)$-configuration ν is a sequence of riggings of the partitions $\nu^{(k)}$ such that for every part of $\nu^{(k)}$ of length i and label x,

$$0 \le x \le P_i^{(k)}(\nu). \tag{5}$$

The pair (ν, J) is called a rigged configuration. The set of riggings of admissible $(\lambda;\mu)$-configurations is denoted by $\mathrm{RC}(\lambda;\mu)$. Let $(\nu, J)^{(k)}$ be the k-th rigged partition of (ν, J). A string $(i, x) \in (\nu, J)^{(k)}$ is said to be singular if $x = P_i^{(k)}(\nu)$, that is, its label takes on the maximum value.

Define the cocharge and charge of $(\nu, J) \in \mathrm{RC}(\lambda; R)$ by

$$\begin{aligned} cc(\nu, J) &= cc(\nu) + |J| \\ c(\nu, J) &= c(\nu) + |J| \\ |J| &= \sum_{k,i \ge 1} |J_i^{(k)}| \end{aligned}$$

where $J_i^{(k)}$ is the partition inside the rectangle of height $m_i(\nu^{(k)})$ and width $P_i^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size i.

3.2 The Bijection

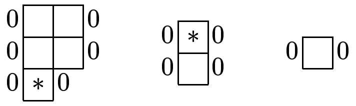
There are several variants of the bijection between tableaux and rigged configurations. Here we use the rowwise quantum number bijection $\psi : \mathrm{CST}(\lambda;\mu) \to \mathrm{RC}(\lambda;\mu)$.

The map from $\mathrm{CST}(\lambda;\mu)$ to $\mathrm{RC}(\lambda;\mu)$ is defined recursively on $|T|$, the number of cells in $T \in \mathrm{CST}(\lambda;\mu)$. The empty tableau maps to the empty rigged configuration. Now start with $T \in \mathrm{CST}(\lambda;\mu)$ with $|T| > 0$. Write T^- for the tableau obtained by removing the rightmost largest letter from T (which occurs in row r, say) and λ^- for the shape of T^-. Let $\mu^- = (\mu_1, \mu_2, \ldots, \mu_{L-1}, \mu_L - 1)$. Since $T^- \in \mathrm{CST}(\lambda^-; \mu^-)$, by induction $\psi(T^-) = (\overline{\nu}, \overline{J})$ is defined. Let $s^{(r)} = \infty$. For $k = r-1$ down to 1, select the longest singular string in $(\overline{\nu}, \overline{J})^{(k)}$ of length $s^{(k)}$ (possibly of zero length) such that $s^{(k)} \leq s^{(k+1)}$. With the convention $s^{(0)} = \mu_L - 1$, it can be shown that $s^{(0)} \leq s^{(1)}$ as well. Then $\psi(T) := (\nu, J)$ is obtained from $(\overline{\nu}, \overline{J})$ by lengthening each of the selected strings by one, resetting their labels to make them singular with respect to the vacancy numbers in the definition of $\mathrm{RC}(\lambda; \mu)$, and leaving all other strings unchanged.

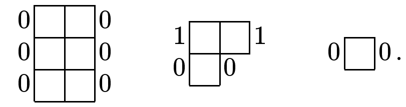
Example 2. Take $\mu = (2, 2, 2, 2, 1)$, $\lambda = (3, 3, 2, 1)$ and

$$T = \begin{array}{l} 1\,1\,3 \\ 2\,2\,4 \\ 3\,5 \\ 4 \end{array} \quad \text{so that} \quad T^- = \begin{array}{l} 1\,1\,3 \\ 2\,2\,4 \\ 3 \\ 4 \end{array}$$

and $r = 3$. The rigged configuration corresponding to T^- is



where the labels are written to the right of each part and the vacancy numbers to the left. The selected strings are indicated by $*$. Hence the rigged configuration corresponding to T is



The following theorem is a variant of [13, Definition-Proposition 4.1 and Theorem 9.1].

Theorem 1. *The bijection $\psi : \mathrm{CST}(\lambda; \mu) \to \mathrm{RC}(\lambda; R)$ is well-defined and preserves the statistics. That is, for $T \in \mathrm{CST}(\lambda; \mu)$ we have $c(T) = c(\psi(T))$.*

An immediate corollary is the following.

Corollary 1. *For λ a partition and μ an array of nonnegative integers*

$$K_{\lambda\mu}(q) = \sum_{(\nu,J)\in \mathrm{RC}(\lambda;\mu)} q^{c(\nu,J)}.$$

3.3 Rigged Configurations and Level-restriction

Here we introduce the most important new definition in this note, namely, that of a level-restricted rigged configuration.

Say that a partition λ is restricted of level ℓ if $\lambda_1 - \lambda_n \leq \ell$, recalling that it is assumed that all partitions have at most n parts, some of which may be zero. For a shape λ restricted of level ℓ define $\tilde{\ell} = \ell - (\lambda_1 - \lambda_n)$, which is nonnegative by assumption.

Set $\lambda' = (\lambda_1 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)^t$ and denote the set of all column-strict tableaux of shape λ' over the alphabet $\{1, 2, \ldots, \lambda_1 - \lambda_n\}$ by $\mathrm{CST}(\lambda')$. Define a table of modified vacancy numbers depending on $\nu \in \mathrm{C}(\lambda; \mu)$ and $t \in \mathrm{CST}(\lambda')$ by

$$P_i^{(k)}(\nu, t) = P_i^{(k)}(\nu) - \sum_{j=1}^{\lambda_k - \lambda_n} \chi(i \geq \tilde{\ell} + t_{j,k}) + \sum_{j=1}^{\lambda_{k+1} - \lambda_n} \chi(i \geq \tilde{\ell} + t_{j,k+1}) \tag{6}$$

for all $i, k \geq 1$, where $\chi(S) = 1$ if the statement S is true and $\chi(S) = 0$ otherwise, and $t_{j,k}$ is the (j,k)-th entry of t. Finally let $x_i^{(k)}$ be the largest part of the partition $J_i^{(k)}$; if $J_i^{(k)}$ is empty set $x_i^{(k)} = 0$.

Definition 1. *Say that $(\nu, J) \in \mathrm{RC}(\lambda; \mu)$ is restricted of level ℓ provided that*

1. *$\nu_1^{(k)} \leq \ell$ for all k.*
2. *There exists a tableau $t \in \mathrm{CST}(\lambda')$, such that for every $i, k \geq 1$,*

$$x_i^{(k)} \leq P_i^{(k)}(\nu, t).$$

Let $\mathrm{C}^\ell(\lambda; \mu)$ be the set of all $\nu \in \mathrm{C}(\lambda; \mu)$ such that the first condition holds, and denote by $\mathrm{RC}^\ell(\lambda; R)$ the set of $(\nu, J) \in \mathrm{RC}(\lambda; \mu)$ that are restricted of level ℓ.

Note in particular that the second condition requires that $P_i^{(k)}(\nu, t) \geq 0$ for all $i, k \geq 1$.

Example 3. Let us consider Definition 1 for two classes of shapes λ more closely:

1. Vacuum case: Let $\lambda = (a^n)$ be rectangular with n rows. Then $\lambda' = \emptyset$ and $P_i^{(k)}(\nu, \emptyset) = P_i^{(k)}(\nu)$ for all $i, k \geq 1$ so that the modified vacancy numbers are equal to the vacancy numbers.
2. Two-corner case: Let $\lambda = (a^\alpha, b^\beta)$ with $\alpha + \beta = n$ and $a > b$. Then $\lambda' = (\alpha^{a-b})$ and there is only one tableau t in $\mathrm{CST}(\lambda')$, namely the Yamanouchi tableau of shape λ'. Since $t_{j,k} = j$ for $1 \leq k \leq \alpha$ we find that

$$P_i^{(k)}(\nu, t) = P_i^{(k)}(\nu) - \delta_{k,\alpha} \max\{i - \tilde{\ell}, 0\}$$

for $1 \leq i \leq \ell$ and $1 \leq k < n$. We wish to thank Anatol Kirillov for communicating this formula to us [14].

Our main result is the following. The proof is given in full detail in [18].

Theorem 2. *The bijection $\psi : \mathrm{CST}(\lambda; \mu) \to \mathrm{RC}(\lambda; \mu)$ restricts to a well-defined bijection $\psi : \mathrm{CST}^\ell(\lambda; \mu) \to \mathrm{RC}^\ell(\lambda; \mu)$.*

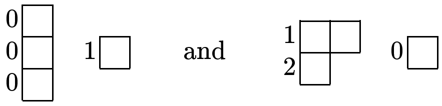
This theorem states that the image of level-restricted tableaux under the bijection are rigged configurations whose riggings lie below certain maxima. In other words, for a given configuration these riggings form an order ideal in the set of all riggings. The maxima are labeled by certain column-strict tableaux (see Definition 1 and equation (6)).

Using the definition in (3), Theorem 1 and Theorem 2 one obtains the following formula for the level-restricted Kostka polynomial:

Theorem 3. *Let ℓ be a positive integer. For λ a partition restricted of level ℓ and μ an array of nonnegative numbers not exceeding ℓ,*

$$K^{\ell}_{\lambda\mu}(q) = \sum_{(\nu,J)\in \mathrm{RC}^{\ell}(\lambda;\mu)} q^{c(\nu,J)}.$$

Example 4. Consider $n=3$, $\ell=2$, $\lambda=(3,2,1)$ and $\mu=(2,1^4)$. Then

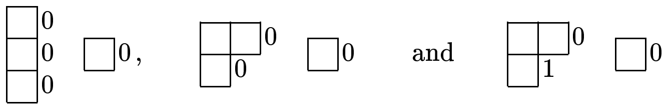


(7)

are in $\mathrm{C}^{\ell}(\lambda;R)$ where again the vacancy numbers are indicated to the left of each part. The set $\mathrm{CST}(\lambda')$ consists of the two elements

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,.$$

Since $\tilde{\ell}=0$ the three rigged configurations



are restricted of level 2 with charges $2,3,4$, respectively. The riggings are given on the right of each part. The last rigged configuration corresponds to the tableau in (2). Hence $K^2_{\lambda\mu}(q)=q^2+q^3+q^4$.

In contrast to this, the Kostka polynomial $K_{\lambda\mu}(q)$ is obtained by summing over both configurations in (7) with all possible riggings below the vacancy numbers. This amounts to $K_{\lambda\mu}(q)=q^2+2q^3+2q^4+2q^5+q^6$.

4 New Expressions for Level-restricted Kostka Polynomials

Let us first briefly review how the fermionic expressions for the Kostka polynomials can be obtained from Corollary 1. Recall that $c(\nu,J)=c(\nu)+|J|$ where $|J|=\sum_{k,i\geq 1}|J^{(k)}_i|$ and $J^{(k)}_i$ is the partition inside the rectangle of height $m_i(\nu^{(k)})$ and width $P^{(k)}_i(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size i. The generating functions of partitions with at most m parts each not exceeding p is given by the q-binomial $\begin{bmatrix} m+p \\ m \end{bmatrix}$ [1, Theorem 3.1]. The q-binomial is defined as

$$\begin{bmatrix} m+p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p}$$

for $m, p \in \mathbb{N}$ and zero otherwise where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$. Hence Corollary 1 implies the following explicit expression for the Kostka polynomials:

Theorem 4. *For λ a partition and μ an array of nonnegative integers*

$$K_{\lambda\mu}(q) = \sum_{\nu \in \mathrm{C}(\lambda;\mu)} q^{c(\nu)} \prod_{k,i \geq 1} \begin{bmatrix} P_i^{(k)}(\nu) + m_i(\nu^{(k)}) \\ m_i(\nu^{(k)}) \end{bmatrix}.$$

Similarly to the Kostka polynomial case, one can rewrite the expression of the level-restricted generalized Kostka polynomials of Theorem 3 in fermionic form.

Lemma 1. *For all $\nu \in \mathrm{C}^\ell(\lambda, \mu)$, $t \in \mathrm{CST}(\lambda')$, $1 \leq k < n$ and $i \geq \ell$ we have $P_i^{(k)}(\nu, t) = 0$.*

This lemma can be proven using [12, (11.2)].

Let $\mathrm{SCST}(\lambda')$ be the set of all nonempty subsets of $\mathrm{CST}(\lambda')$. Furthermore set $P_i^{(k)}(\nu, S) = \min\{P_i^{(k)}(\nu, t) | t \in S\}$ for $S \in \mathrm{SCST}(\lambda')$. Then by inclusion-exclusion the set of allowed rigging for a given configuration $\nu \in \mathrm{C}^\ell(\lambda;\mu)$ is given by

$$\sum_{S \in \mathrm{SCST}(\lambda')} (-1)^{|S|+1} \{J | x_i^{(k)} \leq P_i^{(k)}(\nu, S)\}.$$

Using again the fact that the q-binomial $\begin{bmatrix} m+p \\ m \end{bmatrix}$ is the generating function of partitions with at most m parts each not exceeding p and using $P_\ell^{(k)}(\nu, S) = 0$ by Lemma 1 the level-ℓ restricted generalized Kostka polynomials has the following explicit expression.

Theorem 5.

$$K_{\lambda\mu}^\ell(q) = \sum_{S \in \mathrm{SCST}(\lambda')} (-1)^{|S|+1} \sum_{\nu \in \mathrm{C}^\ell(\lambda;R)} q^{c(\nu)} \prod_{i=1}^{\ell-1} \prod_{k=1}^{n-1} \begin{bmatrix} m_i(\nu^{(k)}) + P_i^{(k)}(\nu, S) \\ m_i(\nu^{(k)}) \end{bmatrix}. \tag{8}$$

In the vacuum case, that is, when $\lambda = ((\frac{|\lambda|}{n})^n)$, the set $\mathrm{SCST}(\lambda')$ only contains $S = \{\emptyset\}$. In this case (8) simplifies to

$$K_{\lambda\mu}^\ell(q) = \sum_{\nu \in \mathrm{C}^\ell(\lambda;R)} q^{c(\nu)} \prod_{i=1}^{\ell-1} \prod_{k=1}^{n-1} \begin{bmatrix} m_i(\nu^{(k)}) + P_i^{(k)}(\nu) \\ m_i(\nu^{(k)}) \end{bmatrix}. \tag{9}$$

When $\mu = (1^{|\lambda|})$ this proves [3, Theorem 1][1]. For general μ equation (9) settles [6, Conjecture 5.7].

5 Branching Functions of Type $\widehat{A}$ and a Conjecture for "Skew" Shapes

The new expressions for the level-restricted Kostka polynomials (8) can be used to derive new expressions for certain type $\widehat{A}$ branching functions by taking appropriate limits. The branching functions are formal power series defined as

$$b^{\Lambda}_{\Lambda',\Lambda''}(q) = \sum_{k\geq 0} q^k c^{\Lambda-k\delta}_{\Lambda',\Lambda''}$$

where $\Lambda, \Lambda', \Lambda''$ are dominant integral weights of levels $\ell' + \ell'', \ell', \ell''$, δ is the null root and $c^{\Lambda-k\delta}_{\Lambda',\Lambda''}$ is the multiplicity of the irreducible highest weight $U_q(\widehat{sl}_n)$-module $V(\Lambda - k\delta)$ in the tensor product $V(\Lambda') \otimes V(\Lambda'')$. Details can be found in ref. [18].

It turns out that the limits of the level-restricted Kostka polynomials only yield the branching functions for $\Lambda'' = \ell''\Lambda_0$ and special choices of Λ'. To get a more complete set of branching functions it is necessary to look at level-restricted Kostka polynomials for "skew" shapes.

It appears that the bijection ψ is not only well-behaved with respect to level-restriction, but also with respect to skew shapes. First evidence of this was given in [11] when the skew shape is λ/ρ, where ρ is rectangular.

For the branching functions "skewing" by ρ corresponds to fixing a subtableau of shape ρ. In fact the choice of the subtableaux does not matter. Computer evidence suggests that under the bijection the effect of a fixed subtableau on the rigged configurations is to impose minima on the set of riggings. Recall that level-restriction imposes maxima on the set of riggings via the modified vacancy numbers.

Here we give an explicit conjecture for the set of minima in the case that the fixed subtableau is the unique tableau T_ρ of shape ρ and content $R_\rho = ((1^{\rho^t_1}), (1^{\rho^t_2}), \ldots)$. Note that this tableau is a Littlewood–Richardson tableau since its content consists of single columns. We denote the set of Littlewood–Richardson tableaux of shape λ and content R by $\mathrm{LRT}(\lambda; R)$. For a definition see [12, 13, 19]. For partitions $\rho \subset \lambda$ define the set of Littlewood–Richardson tableaux with fixed subtableau T_ρ as

$$\mathrm{LRT}(\lambda, \rho, R) = \{T \in \mathrm{LRT}(\lambda, R_\rho \cup R) | T \text{ restricted to shape } \rho \text{ equals } T_\rho\}.$$

Furthermore denote by $\mathrm{RC}(\lambda, \rho, R)$ the image of $\mathrm{LRT}(\lambda, \rho, R)$ under the bijection ψ (for the definition of ψ for a sequence of rectangles see [13]). Set $\rho' = (\rho_1 - \rho_n, \ldots, \rho_{n-1} - \rho_n)^t$. For all $t \in \mathrm{CST}(\rho')$ define the set of minima as

$$M^{(k)}_i(t) = \sum_{j=1}^{\rho_k - \rho_n} \chi(i \leq \rho_1 - \rho_n - t_{j,k}) - \sum_{j=1}^{\rho_{k+1}-\rho_n} \chi(i \leq \rho_1 - \rho_n - t_{j,k+1}).$$

[1] We believe that the proof given in [3] is incomplete.

Conjecture 1. Let λ, ρ be partitions such that $\rho \subset \lambda$ and R a sequence of rectangles. Then $(\nu, J) \in \mathrm{RC}(\lambda, \rho, R)$ if and only if there exists a $t \in \mathrm{CST}(\rho')$ such that $x \geq M_i^{(k)}(t)$ for all x, i, k such that $(i, x) \in (\nu, J)^{(k)}$ and $P_i^{(k)}(\nu) \geq M_i^{(k)}(t)$ for all $i, k \geq 1$.

Note in particular that the second condition is automatically satisfied when $(i, x) \in (\nu, J)^{(k)}$ since $P_i^{(k)}(\nu) \geq x \geq M_i^{(k)}(t)$.

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