# STABILITY MEASURES ON TRIANGULATED CATEGORIES

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Stability measures generalize Bridgeland's stability conditions on a triangulated category. They form a convex cone with the stability conditions among their extreme rays.

Take $\mathcal{C}$ to be a small triangulated category. Let $\mathcal{M}$ be the set of finite positive Borel measures on $\mathbb{R}$ and $\mathcal{M}_{\mathrm{fs}} \subseteq \mathcal{M}$ consist of measures with finite-set support.

**Definition 1.** Write $\mathcal{MC}$ for the maps $\mu\colon \mathrm{Ob}(\mathcal{C}) \to \mathcal{M}$, $A \mapsto \mu_A$ satisfying the following. Write $z_A^\mu(r) = z_A(r) = \int_r^\infty e^{i\pi t} d\mu_A(t) \in \mathbb{C}$ for $r \in \mathbb{R}$; if $\Delta\colon A \to B \to C \to A[1]$ is a distinguished triangle in $\mathcal{C}$, write $z_\Delta^\mu = z_\Delta = z_A - z_B + z_C$.

1. $\mu_{A\oplus B} = \mu_A + \mu_B$,
2. $\mu_{A[1]}(r+1) = \mu_A(r)$,
3. $z_\Delta(r) \in e^{i\pi r} \cdot (\mathbb{H} \cup \mathbb{R}_{\geq 0})$ for any $\Delta$ and any $r \in \mathbb{R}$.

An element $\mu \in \mathcal{MC}$ is called a *stability measure* on $\mathcal{C}$.

Clearly the set $\mathcal{MC}$ is closed under positive linear combinations (thus a convex cone). There is a generalized metric $d$ on $\mathcal{MC}$ defined as

$$d(\mu,\nu) = \sup_{0 \neq A \in \mathcal{C}} \left\{ W(\mu_A/|\mu_A|, \nu_A/|\nu_A|), |\log \frac{|\mu_A|}{|\nu_A|}| \right\} \in [0,\infty], \tag{1}$$

where $|\mu_A| = \int_{\mathbb{R}} d\mu_A \in \mathbb{R}_+$ (the *total mass*) and $W$ is some Wasserstein metric of probability measures (total mass 1).

If $F\colon \mathcal{C} \to \mathcal{D}$ is a triangulated functor, then there is $\mathcal{M}F\colon \mathcal{MD} \to \mathcal{MC}, \mu \mapsto \mu \circ F$, which is functorial. In particular, a stability measure can be restricted to a triangulated subcategory.

**Lemma 2.** *The space $\mathcal{MC}$ carries a (right) action of the group $\widetilde{GL^+}(2,\mathbb{R})$, the universal covering space of $GL^+(2,\mathbb{R})$.*

*Proof.* An element $g \in \widetilde{GL^+}(2,\mathbb{R})$ can be expressed as a pair $(T,f)$ where $f\colon \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(t+1) = f(t)+1$ and $T \in GL^+(2,\mathbb{R})$ such that $f(t) \equiv \mathrm{Im}(\log T(e^{i\pi t}))/\pi \mod 2$. Define $\mu g\colon \mathrm{Ob}(\mathcal{C}) \to \mathcal{M}$ by $(\mu g)_A(t) = |T^{-1}(e^{i\pi t})|\mu_A(f(t))$. Let $s = f(r)$. Then for any distinguished triangle $\Delta$, we have

$$z_\Delta^{\mu g}(s) = T(z_\Delta^\mu(r)) \in T(e^{i\pi r}) \cdot (\mathbb{H} \cup \mathbb{R}_{\geq 0}) = e^{i\pi s} \cdot (\mathbb{H} \cup \mathbb{R}_{\geq 0}).$$

This verifies that $\mu g$ belongs to $\mathcal{MC}$. $\square$

Note that the group $\mathrm{Aut}(\mathcal{C})$ of triangulated auto-equivalences acts on $\mathcal{MC}$ by isometries. It commutes with the action of $\widetilde{GL^+}(2,\mathbb{R})$.

The following definition is due to Bridgeland [Bri07].

**Definition 3.** Write $\mathcal{BC}$ for the pairs $(Z,\mathcal{P})$ of a group homomorphism $Z\colon K(\mathcal{C}) \to \mathbb{C}$ and a map $\mathcal{P}\colon \mathbb{R} \to \mathrm{Sub}(\mathcal{C})$ satisfying the following where $K(\mathcal{C})$ is the Grothendieck group and $\mathrm{Sub}(\mathcal{C})$ is the set of full additive subcategories of $\mathcal{C}$.

1. $Z(X) \in e^{i\pi\phi}\mathbb{R}_+$ for nonzero $X \in \mathcal{P}(\phi)$;
2. $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$;
3. for any $\phi \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_+$, $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\phi+\epsilon), \mathcal{P}(\phi)) = 0$;
4. every $A$ has an associated section $p_A$ of $\mathcal{P}$.

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Here a finitely supported set map $p\colon \mathbb{R} \to \mathrm{Ob}(\mathcal{C})$ is said to be associated to an object $A$ if (taking $\mathbb{R}$ to be the poset category) there is a contravariant functor $c : \mathbb{R} \to \mathcal{C}$ and for each $r$ and sufficiently small $\epsilon > 0$ there is a distinguished triangle $c(r+\epsilon) \to c(r-\epsilon) \to p(r) \to c(r+\epsilon)[1]$ with the first map $c((r-\epsilon, r+\epsilon))$ while for $R$ sufficiently large $c(R) = 0$ and $c(-R) = A$. It is called a section of $P$ if $p(r) \in P(r)$ for any $r \in \mathbb{R}$.

A pair $(Z, \mathcal{P}) \in B\mathcal{C}$ is called a stability condition.

If $\mu \in \mathcal{MC}$, define $Z_\mu(A) = \int_{-\infty}^{\infty} e^{i\pi t} d\mu_A(t)$ and $\mathcal{P}_\mu(\phi) = \{A \mid \mathrm{supp}(\mu_A) = \{\phi\}\}$. Since $Z_\mu(A) = \lim_{r\to-\infty} z_A(r)$ and because of condition (3) of $\mathcal{MC}$, $Z_\mu(A) - Z_\mu(B) + Z_\mu(C) = 0$ for any $\Delta$. Hence $Z_\mu\colon K(\mathcal{C}) \to \mathbb{C}$ is a group homomorphism. Note also that $(Z_\mu, \mathcal{P}_\mu)$ satisfies all but the last condition for $\mathcal{BC}$.

Conversely, if $(Z, \mathcal{P}) \in \mathcal{BC}$, define $\mu_A = \sum_\phi |Z(p_A(\phi))|\delta_\phi$ where $p_A$ is the section of $\mathcal{P}$ associated to $A \in \mathcal{C}$.

**Proposition 4.** *This gives an inclusion $\mathcal{BC} \hookrightarrow \mathcal{MC}$.*

*Proof.* We only need to show $\mu$ is indeed an element in $\mathcal{MC}$. Once this is checked, the injectivity is clear since any $(Z, \mathcal{P}) \in \mathcal{BC}$ is determined by the measures $\mu_E$ for $E \in \mathcal{P}(\phi)$.

For any interval $I \subseteq \mathbb{R}$, define $\mathcal{P}(I) = \{X \in \mathcal{C} \mid \mathrm{supp}(\mu_X) \subseteq I\}$. Then for each $r \in \mathbb{R}$, the subcategory $\mathcal{P}([r,\infty))$ is a bounded t-structure on $\mathcal{C}$ whose heart is $\mathcal{A} = \mathcal{P}([r, r+1))$. Then for $X \in \mathcal{C}$, there is a decomposition $\mu_X = \sum_{i\in\mathbb{Z}} \mu_{H^i_{\mathcal{A}}(X)[-i]}$ into measures supported on successive unit intervals $[r-i, r-i+1)$, where $H^i_{\mathcal{A}}(X) \in \mathcal{A}$ is the $i$-th cohomology of $X$ with respect to the t-structure.

For any $\Delta\colon A \to B \to C \to A[1]$ there is a long exact sequence $H_{\mathcal{A}}(\Delta)$ in $\mathcal{A}$:

$$\cdots \longrightarrow H^i_{\mathcal{A}}(A) \xrightarrow{f^i_A} H^i_{\mathcal{A}}(B) \xrightarrow{f^i_B} H^i_{\mathcal{A}}(C) \xrightarrow{f^i_C} H^{i+1}_{\mathcal{A}}(A) \longrightarrow \cdots .$$

The integral $z_\Delta(r)$ can be expressed as the (finite) sum

$$\begin{aligned} z_\Delta(r) &= \sum_{i\le 0} Z(H^i_{\mathcal{A}}(A)[-i]) - Z(H^i_{\mathcal{A}}(B)[-i]) + Z(H^i_{\mathcal{A}}(C)[-i]) \\ &= \sum_{i\le 0} (-1)^i (Z(\mathrm{Im}\, f^{i-1}_C) + Z(\mathrm{Im}\, f^i_A)) - (-1)^i (Z(\mathrm{Im}\, f^i_A) + Z(\mathrm{Im}\, f^i_B)) \\ &\quad + (-1)^i (Z(\mathrm{Im}\, f^i_B) + Z(\mathrm{Im}\, f^i_C)) \\ &= Z(\mathrm{Im}\, f^0_C). \end{aligned}$$

It belongs to $\in e^{i\pi r} \cdot (\mathbb{H} \cup \mathbb{R}_{\ge 0})$ since $\mathrm{Im}\, f^0_C \in \mathcal{A} = \mathcal{P}([r, r+1))$. □

The generalized metric (1) can be restricted to $\mathcal{BC}$. It then induces the same topology on $\mathcal{BC}$ as the one defined in [Bri07].

**Example 5.** Let $\mathcal{C}$ be the bounded derived category of finite dimensional representations on the quiver $2 \to 1$. Let $\mu_{S_1} \in \mathcal{M}$ be any finite Borel measure. Let $\mu_{S_2}(t+d) = m\mu_{S_1}(t)$ for $0 \le d < 1$ and $m \in \mathbb{R}_+$. Let $\mu_{P_2}(t+d') = m'\mu_{S_1}(t)$ for $0 \le d' < 1$ and $m' \in \mathbb{R}_+$ such that $1 + me^{i\pi d} = m'e^{i\pi d'}$. We claim that this determines a stability measure. In fact, the measure $\mu_{S_1}$ can be approximated by a sequence $\mu^{(k)}_{S_1} \in \mathcal{M}_{\mathrm{fs}}$, $k \in \mathbb{N}$, and define $\mu^{(k)}_{S_2}(t+d) = m\mu^{(k)}_{S_1}(t)$ and $\mu^{(k)}_{P_2}(t+d') = m'\mu^{(k)}_{S_1}(t)$. Note that for each $k$, this defines $\mu^{(k)} \in \mathcal{MC}$ because $\mu^{(k)}$ is a positive linear combination of stability conditions. Then $\mu = \lim_k \mu^{(k)} \in \mathcal{MC}$.

## References

[Bri07] Tom Bridgeland, *Stability conditions on triangulated categories*, Ann. Math. (2) **166** (2007), no. 2, 317–345.