CRITICAL GROUPS OF SOME REGULAR LINE GRAPHS

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Abstract. We describe the critical group of the line graph of the complete bipartite graph $K_{p,q}$. Data is presented that describes the structure of the critical group of some regular line graphs in terms of the critical group of the original graph. This suggests the existence of a large class of regular graphs with such a property.

1. Introduction

The *Laplacian*, $L(G)$, of a loopless graph $G = (V, E)$ is the $|V| \times |V|$ matrix defined by

$$L(G)_{v,v'} = \begin{cases} \deg_G(v) & \text{if } \; v = v', \\ -m_{v,v'} & \text{otherwise,} \end{cases}$$

where $m_{v,v'}$ is the number of edges that have both v and v' as endpoints. Let $\overline{L(G)}^{v,v'}$ be the matrix obtained by striking out row v and column v' from $L(G)$. Define the *critical group* of G, $K(G)$, to be $\mathbb{Z}^{|V|-1}/\mathrm{im}(\overline{L(G)}^{v,v'})$. The critical group is also known an the abelian sandpile group.

We state the following result of Kirchhoff which signifies the importance of the Laplacian.

Theorem 1 (Matrix-Tree Theorem)**.** *For a loopless graph $G = (V, E)$ with $\tau(G)$ spanning trees*

$$\tau(G) = \det(\overline{L(G)}^{v,v'}).$$

Also, if $\lambda_1, \ldots, \lambda_{|V|-1}, 0$ are the eigenvalues of $L(G)$ then

$$\tau(G) = \frac{\lambda_1 \cdots \lambda_{|V|-1}}{|V|}.$$

There are many proofs of this theorem, for example see [1, Theorem 5.4].

This theorem implies $|K(G)| = \tau(G)$. In particular, Cayley's famous formula for the number of spanning trees in the complete graph K_n on n vertices allows us to see that $|K(K_n)| = n^{n-2}$. A simple calculation reveals that $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$. However, this result is misleading in its simplicity, and for many classes of graphs the critical group structure is much more subtle. In this paper we investigate critical group structure of certain regular line graphs.

Given a graph $G = (V, E)$ we define the *line graph* of G, $\mathrm{line}(G)$, to be the graph with vertex set E and an edge set described by the rule that $e, e' \in E$ are adjacent

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in line(G) if e and e' share an endpoint. We are interested in regular line graphs and it is a result of Ray-Chaudhuri [3] that if line(G) is connected and regular then either G is bipartite and vertices in the same bipartition have the same degree, or G is regular. We specialize the former case and characterize the structure of the critical group of line($K_{p,q}$) completely; in the latter case we present some interesting conjectures.

2. Line graphs of complete bipartite graphs

Although the complete bipartite graph is not itself regular, its line graph is; every edge has the same number of edges incident to its endpoints. A particularly nice description of the Laplacian of the line graph of the complete bipartite graph leads to a simple critical group structure.

Theorem 2. *Denote $\mathbb{Z}/a\mathbb{Z}$ by $\mathbb{Z}_a$. For the complete bipartite graph with blocks of size p and q, we have*

$$K(\mathrm{line}(K_{p,q})) \cong \left(\mathbb{Z}_{p(p+q)}\right)^{p-2} \oplus \left(\mathbb{Z}_{q(p+q)}\right)^{q-2} \oplus \left(\mathbb{Z}_{p+q}\right)^{(p-2)(q-2)+1}.$$

In terms of the Laplacian of $K_{p,q}$ the above theorem gives a diagonal matrix that is equivalent to $L(\mathrm{line}(K_{p,q}))$ over the integers. In many cases, we wish this diagonal form to have the property that a particular entry divides all entries that follow it, that is, we want the Smith normal form of $L(\mathrm{line}(K_{p,q}))$. The entries of a matrix written in Smith normal form are called the *invariant factors* of the matrix. As the point of the above theorem is to render $L(\mathrm{line}(K_{p,q}))$ in any $\mathbb{Z}$-equivalent diagonal form, we will be content to state its (non-zero, non-one) invariant factors without proof. Assume that $p \geq q$.

Invariant factor	Multiplicity
$p+q$	$(p-2)(q-2)+1$
$\gcd(p,q)(p+q)$	$q-2$
$p\,(p+q)$	$p-q$
$\mathrm{lcm}(p,q)(p+q)$	$q-2$

Before the proof of the theorem we give some notation and a lemma.

Definition 3. [2] *If M and T are square matrices of the same size, $H_n(M,T)$ is the $n \times n$ (block) matrix defined by*

$$H_n(M,T) := \begin{bmatrix} M-T & -T & \cdots & -T \\ -T & M-T & \ddots & \vdots \\ \vdots & \ddots & \ddots & -T \\ -T & \cdots & -T & M-T \end{bmatrix}.$$

Lemma 4. [2] *$H_n(M,T)$ has the following property:*

$$H_n(M,T) \sim M^{\oplus(n-2)} \oplus \begin{bmatrix} M & T \\ 0 & M-nT \end{bmatrix},$$

where $A \sim B$ means that A and B are equivalent matrices over $\mathbb{Z}$.

Proof. The proof is simple row and column operations, analogous to those which reduced $L(K_n)$ to $n^{\oplus(n-2)} \oplus 1 \oplus 0$. □

In the particular case when is T the identity matrix I we get

$$H_n(M, I) \sim M^{\oplus(n-2)} \oplus I \oplus M(M - nI).$$

This follows since for any two matrices A, B we have

$$\begin{bmatrix} A & I \\ 0 & B \end{bmatrix} \sim I \oplus AB.$$

Proof of Theorem 2. We begin by describing the Laplacian of $\text{line}(K_{p,q})$. Label vertices in the block of size p by $v_1, \ldots, v_p$, and those in the block of size q by $u_1, \ldots, u_q$. Labels the edges of $K_{p,q}$ so that for $e_1, \ldots, e_q$ the edge e_i connects v_1 to u_i. Continue this edge labeling process at each v_i, labeling the edges $e_{iq+1}, \ldots, e_{iq+q}$ so that e_{iq+j} connects vertex v_i to u_j. This labeling describes $L(\text{line}(K_{p,q}))$ as the $p \times p$ block matrix with

$$q\left\{\begin{bmatrix} p+q-2 & -1 & \cdots & -1 \\ -1 & p+q-2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p+q-2 \end{bmatrix}\right. \tag{1}$$

on the diagonal and $-I_q$ on the off-diagonal. This makes sense since the diagonal block entry describes edges that meet at a common vertex, hence these edges form a clique in the line graph. Also, the off diagonal block entry describes edges emanating from different vertices, and two edges from different vertices only meet when they are the i-th edge from their respective vertex.

Note that the diagonal entry of (1) can be written as $H_q(p+q, 1) - I_q$. This leads to the following compact description of $L(\text{line}(K_{p,q}))$:

$$L(\text{line}(K_{p,q})) = H_p(H_q(p+q, 1), I_q).$$

We now apply Lemma 4 to see

$$L(\text{line}(K_{p,q})) \sim$$
$$H_q(p+q, 1)^{\oplus(p-2)} \oplus I_q \oplus (H_q(p+q, 1) - pI_q)H_q(p+q, 1)$$

It is immediate from the definition that $H_q(p+q, 1) - pI_q = H_q(q, 1)$ so we write

$$\begin{aligned} &L(\text{line}(K_{p,q})) \\ &\sim H_q(p+q, 1)^{\oplus(p-2)} \oplus I_q \oplus H_q(p+q, 1)H_q(q, 1) \\ &\sim (p+q)^{\oplus(p-2)(q-2)} \oplus 1^{\oplus(p-2)} \oplus I_q \oplus (p(p+q))^{\oplus(p-2)} \oplus H_q(p+q, 1)H_q(q, 1) \end{aligned} \tag{2}$$

Now, the i, j entry of $H_n(M, I)H_n(M', I)$ is

$$(M - I)(M' - I) + (n-1)I = MM' - (M + M' - nI),$$

if $i = j$ and

$$-M + I - M' - I + (n-2)I = -(M + M' - nI),$$

otherwise. Thus $H_q(p+q, 1)H_q(q, 1) = H_q(q(p+q), p+q))$ and

$$\begin{aligned} H_q(q(p+q), p+q)) &\sim (q(p+q))^{\oplus(q-2)} \begin{bmatrix} q(p+q) & (p+q) \\ 0 & 0 \end{bmatrix} \\ &\sim (q(p+q))^{\oplus(q-2)} \oplus (p+q) \oplus 0. \end{aligned} \tag{3}$$

From (2) and (3) we can write

$$K(\mathrm{line}(K_{p,q})) \cong \left(\mathbb{Z}_{p(p+q)}\right)^{p-2} \oplus \left(\mathbb{Z}_{q(p+q)}\right)^{q-2} \oplus \left(\mathbb{Z}_{p+q}\right)^{(p-2)(q-2)+1}.$$

3. Spanning trees in line graphs

The motivation for the conjectures in the next section can been found in the following theorem.

Theorem 5. *Let $G = (V, E)$ be a connected d-regular loopless graph. Then*

$$\tau(\mathrm{line}(G)) = 4(2d)^{\beta(G)-2}\tau(G),$$

where $\beta(G) = |E| - |V| + 1$ is the number of independent cycles in G.

This theorem follows from the following lemma which simply relates the Laplacian of a graph to that of its line graph.

Lemma 6. *Let $G = (V, E)$ be a d-regular loopless graph. Let $X(G)$ be the $|V| \times |E|$ unsigned incidence matrix of G with $X(G)_{v,e} = 1$ if e is incident to v and $X(G)_{v,e} = 0$ otherwise. Then*

$$L(G) = 2dI_{|V|} - X(G)X(G)^t$$
$$L(\mathrm{line}(G)) = 2dI_{|E|} - X(G)^tX(G)$$

Proof. The i, j entry of $X(G)X(G)^t$ is the inner product of the v_i row and the v_j row. This is d if $i = j$ and the number of edges connecting v_i to v_j if $i \neq j$. Thus

$$(X(G)X(G)^t)_{i,j} + L(G)_{i,j} = \begin{cases} 2d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The i, j entry of $X(G)^tX(G)$ is the inner product of the e_i column and the e_j column. This is 2 if e_i and e_j share both endpoints, 1 if they share one endpoint, and 0 if the share no endpoints. The i, j entry of $L(\mathrm{line}(G))$ is $2(d-1)$ on the diagonals and minus the number of endpoints that e_i and e_j share on the off diagonals. Hence

$$(X(G)^tX(G)^t)_{i,j} + L(\mathrm{line}(G))_{i,j} = \begin{cases} 2 + 2(d-1) = 2d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of Theorem 5. It is a result from linear algebra that for two matrices A and B of sizes $n \times m$ and $m \times n$, respectively, AB has the same eigenvalues as BA, with the addition of $m - n$ eigenvalues of 0. From the lemma and this we can write

$$(2d - \lambda)^{|E|-|V|} \det(L(G) - \lambda I_{|V|}) = \det(L(\mathrm{line}(G)) - \lambda I_{|E|}).$$

Hence $L(\mathrm{line}(G))$ has the same eigenvalues as $L(G)$ with addition of $|E| - |V|$ eigenvalues equal to $2d$. Writting the eigenvalues of $L(G)$ as $\lambda_1 \cdots \lambda_{|V|-1}, 0$ and applying Theorem 1 to $\mathrm{line}(G)$ we get

$$\begin{aligned} \tau(\mathrm{line}(G)) &= (2d)^{|E|-|V|} \frac{\lambda_1 \cdots \lambda_{|V|-1}}{|E|} \frac{|V|}{|V|} \\ &= (2d)^{|E|-|V|} \frac{\tau(G)}{|V|d/2} |V| \\ &= 4(2d)^{\beta(G)-2}\tau(G). \end{aligned}$$

□

4. Conjectures

Consider the following example.

Example 7. The critical group of the Petersen graph is isomorphic to $\mathbb{Z}_{10}^3 \oplus \mathbb{Z}_2$, and the critical group of its line graph is isomorphic to $\mathbb{Z}_{60}^3 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2^2$. The Petersen graph has $\beta = 6$ and is 3-regular. Theorem 5 might suggest that the latter group was obtained from the former by multiplying the 10's and the 2 by $2 \cdot 3 = d \cdot \beta$ and then adding in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as thus prescribed.

This phenomenon is not unique as we shall now discuss.

Take G to be a d-regular graph ($d > 2$) and $L(G)$ to have non-zero/one invariant factors $d_1, \ldots, d_r$ so that $K(G) \cong \bigoplus_{i=1}^r \mathbb{Z}_{d_i}$. Then suppose that $L(\text{line}(G))$ has invariant factors which include $2dd_1, \ldots, 2dd_r$. Theorem 5 shows that the product of the remaining invariant factors must be $4(2d)^{\beta(G)-r-2}$. We might hope that the left over invariant factors are $2d$ occurring $\beta(G) - r - 2$ times and two 2's, i.e., that

$$K(\text{line}(G)) \cong \left(\bigoplus_{i=1}^{r} \mathbb{Z}_{2dd_i} \right) \oplus \mathbb{Z}_{2d}^{\beta(G)-r-2} \oplus \mathbb{Z}_2^2.$$

When this occurs say that G obeys *the naive prediction*. Computer evidence has shown many regular graphs obey the naive prediction. Consider the following family of examples.

In [2] the structure of the critical group of $K_{m_1,\ldots,m_k}$ was completely described. When all the $m_i = m > 1$ and $k > 2$ the result was particularly simple:

$$K(K_{\underbrace{m,\ldots,m}_{k}}) \cong \mathbb{Z}_{k-1} \oplus \left(\mathbb{Z}_{(k-1)m}\right)^{k(m-2)} \oplus \mathbb{Z}_{(k-1)m^2} \oplus \left(\mathbb{Z}_{k(k-1)m^2}\right)^{k-2}$$

Computer evidence suggested the following conjecture.

$$K(\text{line}(K_{\underbrace{m,\ldots,m}_{k}})) \cong \mathbb{Z}_{2d(k-1)} \oplus \left(\mathbb{Z}_{2d(k-1)m}\right)^{k(m-2)} \oplus \mathbb{Z}_{2d(k-1)m^2} \oplus \left(\mathbb{Z}_{2dk(k-1)m^2}\right)^{k-2} \oplus (\mathbb{Z}_{2d})^{\gamma} \oplus A_4$$

where

$$A_4 = \begin{cases} \mathbb{Z}_4 & \text{if both } k \text{ and } m \text{ are odd,} \\ \mathbb{Z}_2^2 & \text{otherwise,} \end{cases}$$

$$d = (k-1)m = \text{ vertex-regularity of } K_{\underbrace{m,\ldots,m}_{k}},$$

$$\gamma = \binom{k}{2} m^2 - km - k(m-2) - (k-2) - 1.$$

As we hoped,

$$\gamma = \beta - (k(m-2) + (k-2) + 2) - 2.$$

There are, however, a number of counterexamples to the naive prediction. For instance the hypercubes in dimension $3, 4$, and 5 fail to fail to obeythe naive prediction.

All 3-regular graphs on 10 vertices[1] obey the naive prediction except the two shown in Figure 1.

FIGURE 1. Counterexamples

The graph on the left has critical group isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{24}$ while its line graph has critical group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_6^3 \oplus \mathbb{Z}_{144}^2$. The graph on the right has critical group isomorphic to $\mathbb{Z}_{11} \oplus \mathbb{Z}_{165}$ while its line graph has critical group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_6^3 \oplus \mathbb{Z}_{66} \oplus \mathbb{Z}_{330}$.

It is tempting to place simple restrictions on a regular graph G, such as planarity, and hope that it obeys the naive prediction. In light of the fact that the hypercube in dimension 3 is planar, such a condition does not give the desired result. One can be even more stringent and consider outerplanar regular graphs (planar graphs in which every vertex can be drawn on the outerface), but computer evidence suggests that these graphs *never* obey the naive prediction. Finally, any condition of planarity is far too restrictive since $K_{\underbrace{m,\dots,m}_{k}}$ is non-planar for $k, m \geq 3$.

Thus the question of what properties does a regular graph G need to have for it to obey the naive prediction is subtle and remains unsolved.

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[1] These graphs were obtained using the computer program GENREG, see [4].