

On Explicit Formula for Restricted Partition Function

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Abstract

A new recursive procedure of the calculation of a restricted partition function is suggested. An explicit combinatorial formula for the restricted partition function is found based on this procedure.

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1 Introduction

The problem of partitions of positive integers has long history started from the work of Euler [1] who "laid a foundation of the theory of partitions" [2], introducing the idea of generating functions. Many great mathematicians, like Cayley, Sylvester, MacMahon, Ramanujan, and others contributed to the development of the theory, using Euler idea.

Cayley [3] found explicit formulas for number $p_k(n)$ of partitions of positive integer n into at most k parts with small k . He also suggested a method of decomposition of the corresponding generating function

$$G_k(t) = \prod_{i=1}^k \frac{1}{1-t^i} = \sum_{n=0}^{\infty} p_k(n) t^n,$$

and gave the combinatorial formula for such decompositions (unfortunately, this formula itself requires knowledge of all partitions of k).

Sylvester was the next mathematician who provided a new insight and made a remarkable progress in this field. He introduced [4] the so-called Ferrers graphs for presentation of partitions. He also found [5, 6] the procedure enabling to determine a *restricted* partition functions, and described symmetry features of such functions. The restricted partition function $p(n, \mathbf{d}^m) \equiv p(n, \{d_1, d_2, \dots, d_m\})$ is a number of partitions of n into positive integers $\{d_1, d_2, \dots, d_m\}$, each not greater than n . It is very simple to show that the generating function in this case takes the form

$$G(\mathbf{d}^m, t) = \prod_{i=1}^m \frac{1}{1-t^{d_i}} = \sum_{n=0}^{\infty} p(n, \mathbf{d}^m) t^n. \quad (1)$$

Sylvester showed that the restricted partition function may be presented as a sum of "waves", each wave closely related to prime roots of unit of degree n , where n are prime divisors of elements of the set \mathbf{d}^m . This fact was known to Herschel [7] who introduced a notion of *circulator* and Cayley who used its elegant version called *prime circulator* (see [8] for more information). Namely, Sylvester showed that each wave W_i , where i runs over distinct factors in d_1, d_2, \dots, d_m , is a coefficient of t^{-1} in the series expansion in ascending powers of t of

$$e^{sw_k} \prod_{r=1}^m \frac{1}{1 - e^{d_r u_k}} \quad , \quad w_k = 2\pi i \frac{p_k}{q} + t \quad , \quad u_k = 2\pi i \frac{p_k}{q} - t \quad , \quad (2)$$

and $p_1, p_2, \dots, p_{max\ k}$ are integers (unity included) smaller than i and prime to it. It should be noted here that the above result is only a recipe for calculation of the partition function and doesn't provide an explicit formula.

Sylvester found [6] that the shifted partition function

$$q(n, \mathbf{d}^m) \equiv p\left(n - \frac{1}{2} \sum_{i=1}^m d_i, \mathbf{d}^m\right)$$

has following parity properties:

$$q(n, \mathbf{d}^{2m}) = -q(-n, \mathbf{d}^{2m}), \quad q(n, \mathbf{d}^{2m+1}) = q(-n, \mathbf{d}^{2m+1}),$$

and established that these functions have zeros at all integer values of n from 0 to $m/2 - 1$ for even m and at all semiinteger values from $1/2$ to $m/2 - 1$ for odd m . He suggested to use knowledge of partition function zeroes for its construction using the method of indeterminate coefficients.

Recently, a different presentation of the partition function, which may be called *polynomial expansion*, was introduced in [9], where a new recursive procedure for calculation of the restricted partition function is found. It permits to reconstruct in an unified way nearly all terms of expansion, except one, which also demands usage of the method of indeterminate coefficients.

In this article I present an approach enabling to overcome this inconsistency, and to determine all terms of expansion in the framework of a single recursive procedure. This method also produces the explicit formula of the restricted partition function.

2 Polynomial part of the partition function

J.J. Sylvester showed that the restricted partition function may be written as a sum of "waves", he found the recipe for calculation of each such wave. We consider in this section purely polynomial part of the partition function which corresponds to the wave W_1 . It may be found as a coefficient of t^{-1} in an expansion of the generator

$$G_1(s, t) = \frac{e^{st}}{\prod_{i=1}^m (1 - e^{-d_i t})}. \quad (3)$$

Sylvester found that $W_1(s)$ depends on Bernoulli numbers and sums of powers of elements d_i . We find an explicit form of polynomial part $W_1(s)$ through the Bernoulli polynomials of higher

order. We start from the generating function for the Bernoulli polynomials of higher order $B_n^{(m)}(s|d_1, d_2, \dots, d_m)$ [10]:

$$\left(\prod_{i=1}^m d_i\right) t^m \frac{e^{st}}{\prod_{i=1}^m (e^{d_i t} - 1)} = \sum_{n=0}^{\infty} B_n^{(m)}(s|\mathbf{d}^m) \frac{t^n}{n!}, \quad (4)$$

where we use a shortcut notation

$$\mathbf{d}^m = \{d_1, d_2, \dots, d_m\}.$$

One immediately obtains a presentation of W_1 generator in the form

$$G_1(s, t) = \frac{1}{\pi(\mathbf{d}^m)t^m} \sum_{n=0}^{\infty} B_n^{(m)}(s|\mathbf{d}^m) \frac{t^n}{n!}, \quad (5)$$

where

$$\pi(\mathbf{d}^m) = \prod_{i=1}^m d_i.$$

The coefficient of $1/t$ in the above expression is given by the term with $n = m - 1$

$$W_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} B_{m-1}^{(m)}(s|\mathbf{d}^m). \quad (6)$$

It is useful to consider a *shifted* restricted partition function defined as follows

$$V(s, \mathbf{d}^m) = W(s - \xi(\mathbf{d}^m), \mathbf{d}^m), \quad \xi(\mathbf{d}^m) = \frac{1}{2} \sum_{i=1}^m d_i, \quad (7)$$

and its polynomial part $V_1(s, \mathbf{d}^m)$ which is cast in

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} B_{m-1}^{(m)}(s - \xi(\mathbf{d}^m)|\mathbf{d}^m). \quad (8)$$

Now we may use formula for Bernoulli polynomials of higher order found by Nörlund [11]

$$B_n^{(m)}(s|\mathbf{d}^m) = B_n^{(m)}\left(s + \sum_{i=1}^m d_i|\mathbf{d}^m\right)$$

to arrive at

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} B_{m-1}^{(m)}(s + \xi(\mathbf{d}^m)|\mathbf{d}^m). \quad (9)$$

Then we apply another formula

$$B_n^{(m)}(s|\mathbf{d}^m) = \sum_{l=0}^n C_n^l \frac{D_l^{(m)}(\mathbf{d}^m)}{2^l} (s - \xi(\mathbf{d}^m))^{n-l}$$

where $D_l^{(m)}(\mathbf{d}^m) \equiv 2^l B_l^{(m)}(\xi(\mathbf{d}^m)|\mathbf{d}^m)$ and $D_{2k+1}^{(m)}(\mathbf{d}^m) = 0$ and obtain

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \frac{D_l^{(m)}(\mathbf{d}^m)}{2^l}. \quad (10)$$

The quantities $D_n^{(m)}(\mathbf{d}^m)$ can be calculated using a recursive relation

$$D_n^{(m)}(\mathbf{d}^m) = \sum_{l=0}^n C_n^l d_m^l D_l D_{n-l}^{(m-1)}(\mathbf{d}^{m-1}), \quad (11)$$

where D_l is expressed through the value of Bernoulli polynomial of order l at fixed value of argument.

$$D_l = 2^l B_l(1/2).$$

Instead of (11) one may use more symmetric form

$$D_n^{(m)}(\mathbf{d}^m) = \sum C_n^{\mathbf{r}} \prod_{i=1}^m d_i^{r_i} D_{r_i}, \quad (12)$$

where

$$C_n^{\mathbf{r}} = \frac{n!}{\prod_{i=1}^m r_i!}$$

is a multinomial coefficient and summation in (12) is performed over all r_i such that

$$\sum_{i=1}^m r_i = n.$$

The resulting symmetric expression for the polynomial part of the shifted restriction function is

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \sum C_l^{\mathbf{r}} \prod_{i=1}^m d_i^{r_i} B_{r_i}(1/2). \quad (13)$$

The inner sum in the above expression can be rewritten using the symbolic notation accepted in theory of Bernoulli polynomials (see, for example, [11]):

$$\sum_{i=1}^l C_l^{\mathbf{r}} \prod_{i=1}^m d_i^{r_i} B_{r_i}(1/2) \equiv \left(\sum_{i=1}^m d_i {}^i B(1/2) \right)^l, \quad (14)$$

where powers r_i of ${}^i B(1/2)$ are converted into indices

$${}^i B^{r_i}(1/2) \Rightarrow B_{r_i}(1/2).$$

Using this notation one immediately arrives at the compact expression of $V_1(s, \mathbf{d}^m)$

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} \left(s + \sum_{i=1}^m d_i {}^i B(1/2) \right)^{m-1}. \quad (15)$$

Finally, using (7), we return to the expression for $W_1(s, \mathbf{d}^m)$

$$W_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi(\mathbf{d}^m)} \left(s + \sum_{i=1}^m d_i \left[1/2 + {}^iB(1/2) \right] \right)^{m-1}. \quad (16)$$

It is easily checked that the expression (15) verifies the general recursive relation for the shifted restriction function (which is valid also for its polynomial part):

$$V(s, \mathbf{d}^m) - V(s - d_m, \mathbf{d}^m) = V\left(s - \frac{d_m}{2}, \mathbf{d}^{m-1}\right). \quad (17)$$

Introducing the power expansion of V_1 in the form

$$V_1(s, \mathbf{d}^m) = \sum_{j=1}^m R_j^m s^{m-j},$$

one may write for the coefficients R_j^m

$$R_j^m = \frac{C_{m-1}^{j-1}}{(m-1)!\pi(\mathbf{d}^m)} \left(\sum_{i=1}^m d_i {}^iB(1/2) \right)^{j-1}. \quad (18)$$

The recursion relation (11) is equivalent to the following recursion ($1 \leq j < m$):

$$R_j^m = \frac{1}{m-j} \sum_{l=0}^{j-1} d_m^{l-1} C_{m-1-j+l}^l B_l(1/2) R_{j-l}^{m-1}. \quad (19)$$

It should be noted an important distinction between two last expressions – the formula (18) is the explicit expression for the polynomial part of the shifted restriction function, while the recursion presents an imcomplete procedure (note that j in (19) cannot be set equal to m , so that R_m^m is not defined in the framework of this procedure).

The recursion (19) is a consequence of (17), which repeated usage leads to more general recursive relation

$$V(s + \tau_m, \mathbf{d}^m) = V(s, \mathbf{d}^m) + \sum_{p=0}^{\delta_m-1} V(s + \tau_m - \lambda_p \cdot d_m, \mathbf{d}^{m-1}), \quad (20)$$

where

$$\lambda_p = p + 1/2, \quad \delta_m = \tau_m/d_m,$$

and

$$\tau_m \equiv \tau(\mathbf{d}^m) = \text{LCM}(\mathbf{d}^m),$$

where $\text{LCM}(\mathbf{d}^m)$ denotes a least common multiple of the set \mathbf{d}^m . The function $V(s, \mathbf{d}^m)$ might also be written in a "polynomial" form

$$V(s, \mathbf{d}^m) = \sum_{j=1}^m R_j^m(s) s^{m-j}, \quad (21)$$

which leads to a recursive formula for τ_m -periodic function $R_j^m(s)$ for $1 \leq j < m$ (see [9])

$$R_j^m(s) = \frac{1}{m-j} \cdot \sum_{l=0}^{j-1} d_m^{l-1} C_{m-1-j+l}^l \cdot \delta_m^{l-1} \sum_{p=0}^{\delta_m-1} B_l\left(1 - \frac{\lambda_p d_m}{\tau_m}\right) \cdot R_{j-l}^{m-1}(s - \lambda_p \cdot d_m). \quad (22)$$

Introducing the shift operator

$$\mathbf{S}(s, \Delta) : \{\mathbf{S}(s, \Delta)f(s) = f(s - \Delta)\}, \quad (23)$$

we rewrite the above recursion in the form

$$R_j^m(s) = \frac{1}{m-j} \cdot \sum_{l=0}^{j-1} d_m^{l-1} C_{m-1-j+l}^l \cdot \left[\delta_m^{l-1} \sum_{p=0}^{\delta_m-1} B_l\left(1 - \frac{\lambda_p d_m}{\tau_m}\right) \mathbf{S}(s, \lambda_p d_m) \right] \cdot R_{j-l}^{m-1}(s). \quad (24)$$

Comparison of (24) with (19) suggests that a replacement

$${}^i B_{r_i}(1/2) \implies {}^i \mathbf{B}_{r_i} = (\tau_i/d_i)^{r_i-1} \sum_{p_i=0}^{\tau_i/d_i-1} B_{r_i}\left(1 - \frac{\lambda_{p_i} d_i}{\tau_i}\right) \mathbf{S}(s, \lambda_{p_i} d_i) \quad (25)$$

may be useful in transition from the formulas for the polynomial part of the partition function to those of for the function itself.

Setting in (24) all R_j^m independent of s and using the multiplication theorem for the Bernoulli polynomials [10]

$$\sum_{r=0}^{m-1} B_n\left(x + \frac{r}{m}\right) = m^{-(n-1)} B_n(mx),$$

we immediately reproduce (19). The recursion (22) fails to produce $R_m^m(s)$, nevertheless partial information about it can be extracted. It is useful to separate $R_m^m(s)$ into two terms

$$R_m^m(s) = \mathcal{R}_m^m(s) + r_m^m(s), \quad (26)$$

where

$$\mathcal{R}_m^m(s) = \sum_{l=1}^{m-1} \frac{\tau_m^{l-1}}{l} \sum_{p=0}^{\delta_m-1} B_l\left(1 - \frac{\lambda_p d_m}{\tau_m}\right) \cdot R_{m-l}^{m-1}(s - \lambda_p \cdot d_m), \quad (27)$$

is τ_{m-1} -periodic function, and the other term, d_m -periodic function,

$$r_m^m(s) = r_m^m(s - d_m)$$

remains indeterminate. In [9] $r_m^m(s)$ is found by applying the method of indeterminate coefficients using knowledge of partition function zeroes. It can be checked using (21) that the partition function may be written also as

$$V(s, \mathbf{d}^m) = r_m^m(s) + \sum_{l=1}^{m-1} \frac{\tau_m^{l-1}}{l} \sum_{p=0}^{\delta_m-1} B_l\left(\frac{s + \lambda_p d_m}{\tau_m}\right) R_{m-l}^{m-1}(s + \lambda_p d_m). \quad (28)$$

The corresponding expression for the polynomial part of the partition function reads

$$V_1(s, \mathbf{d}^m) = r_m^m + \sum_{l=1}^{m-1} \frac{d_m^{l-1}}{l} B_l \left(\frac{1}{2} + \frac{s}{d_m} \right) R_{m-l}^{m-1}. \quad (29)$$

Setting in the above expression $s = 0$ we obtain polynomial analog of (26)

$$R_m^m = r_m^m + \sum_{l=1}^{m-1} \frac{d_m^{l-1}}{l} B_l \left(\frac{1}{2} \right) R_{m-l}^{m-1}. \quad (30)$$

Substituting into it the general expression (18) we determine r_m^m

$$r_m^m = \frac{1}{(m-1)! \pi(\mathbf{d}^m)} \left(\sum_{i=1}^{m-1} d_i {}^i B(1/2) \right)^{m-1}. \quad (31)$$

3 Calculation of $r_m^m(s)$

Our goal is to find yet indeterminate $r_m^m(s)$ using (31) and recursive relation (17) for the partition function and its polynomial part and its corrolaries (19) and (22).

In a simplest case $m = 1$ it is easily checked that the shifted partition function $V(s, \{d_1\})$ is a d_1 -periodic function given by

$$V(s, \{d_1\}) = R_1^1(s) = r_1^1(s) = \Psi_{d_1}(s - \frac{d_1}{2}) = \sum_{p_1=0}^{\tau_1/d_1-1} B_0\left(\frac{\lambda_{p_1} d_1}{\tau_1}\right) \Psi_{d_1}(s - \lambda_{p_1} d_1), \quad (32)$$

Here $\tau_1 \equiv d_1$, so that p_1 takes only zero value; we also use parity property of Bernoulli polynomials $B_0(1-x) = B_0(x) \equiv 1$. The periodic function $\Psi_{d_1}(s)$ is defined as a sum of prime roots of unit of degree d_1 :

$$\Psi_{d_1}(s) = \frac{1}{d_1} \sum_{k=0}^{d_1-1} \exp\left(\frac{2\pi i k s}{d_1}\right) = \begin{cases} 1, & s = 0 \pmod{d_1} \\ 0, & s \neq 0 \pmod{d_1} \end{cases}$$

The polynomial part of this function

$$V_1(s, \{d_1\}) = \frac{1}{d_1},$$

this also follows from (15) for $m = 1$.

Consider $m = 2$, and find $R_1^2(s)$ using the recursive relation (22) for the set $\{d_1, d_2\}$

$$\begin{aligned} R_1^2(s) &= \frac{1}{\tau_2} \sum_{p_2=0}^{\tau_2/d_2-1} B_0\left(1 - \frac{\lambda_{p_2} d_2}{\tau_2}\right) R_1^1(s - \lambda_{p_2} d_2) = \\ &= \frac{1}{\tau_2} \sum_{p_2=0}^{\tau_2/d_2-1} B_0\left(1 - \frac{\lambda_{p_2} d_2}{\tau_2}\right) \sum_{p_1=0}^{\tau_1/d_1-1} B_0\left(1 - \frac{\lambda_{p_1} d_1}{\tau_1}\right) \Psi_{d_1}(s - \lambda_{p_2} d_2 - \lambda_{p_1} d_1). \end{aligned} \quad (33)$$

The symmetry of the problem w.r.t. the permutations of the set elements implies that one can apply an interchange $d_1 \leftrightarrow d_2$ in order to produce another valid form of $R_1^2(s)$. The corresponding polynomial part is found as

$$R_1^2 = \frac{{}^1B_0(1/2) {}^2B_0(1/2)}{d_1 d_2}.$$

It is clear that (33) can be produced from the above as follows – we replace the fraction $1/d_1$ by its counterpart $\Psi_{d_1}(s - d_1/2)$ and then use the replacement (25). The another possible form is found by the replacement $1/d_2 \Rightarrow \Psi_{d_2}(s - d_2/2)$ and application of (25) to it. The free term $R_2^2(s)$ is constructed in two steps. At first we find $\mathcal{R}_2^2(s)$ assuming d_1 the first element of the set. Then (27) gives

$$\begin{aligned} \mathcal{R}_2^2(s) &= \sum_{p_2=0}^{\tau_2/d_2-1} B_1\left(1 - \frac{\lambda_{p_2} d_2}{\tau_2}\right) R_1^1(s - \lambda_{p_2} d_2) = \\ &\sum_{p_2=0}^{\tau_2/d_2-1} B_1\left(1 - \frac{\lambda_{p_2} d_2}{\tau_2}\right) \sum_{p_1=0}^{\tau_1/d_1-1} B_0\left(1 - \frac{\lambda_{p_1} d_1}{\tau_1}\right) \Psi_{d_1}(s - \lambda_{p_2} d_2 - \lambda_{p_1} d_1). \end{aligned} \quad (34)$$

The other part – d_2 -periodic function $r_2^2(s)$ may be produced in this case by the interchange $d_1 \leftrightarrow d_2$ in the above expression, which corresponds to choice of d_2 as the first element of the set.

$$\begin{aligned} r_2^2(s) &= \sum_{p_1=0}^{\tau_2/d_1-1} B_1\left(1 - \frac{\lambda_{p_1} d_1}{\tau_2}\right) R_1^1(s - \lambda_{p_1} d_1) = \\ &\sum_{p_1=0}^{\tau_2/d_1-1} B_1\left(1 - \frac{\lambda_{p_1} d_1}{\tau_2}\right) \sum_{p_2=0}^{\tau_1/d_2-1} B_0\left(1 - \frac{\lambda_{p_2} d_2}{\tau_1}\right) \Psi_{d_2}(s - \lambda_{p_1} d_1 - \lambda_{p_2} d_2). \end{aligned} \quad (35)$$

The corresponding polynomial parts reads

$$r_2^2 = \frac{{}^1B_1(1/2) d_1}{d_1 d_2}$$

and is equal to zero. Nevertheless, we may use the above expression in order to produce (35) – we start from the replacement $1/d_2 \Rightarrow \Psi_{d_2}(s - d_2/2)$ and then apply (25) to arrive at d_2 -periodic function.

We conjecture the following formula for d_m -periodic function $r_m^m(s)$ using its polynomial part (31) which we rewrite in the expanded form

$$r_m^m = \frac{1}{d_m} \cdot \frac{1}{(m-1)!} \sum_{\mathbf{r}}^{m-1} C_{m-1}^{\mathbf{r}} \prod_{i=1}^m d_i^{r_i-1} {}^iB_{r_i}(1/2). \quad (36)$$

We start with $1/d_m \Rightarrow \Psi_{d_m}(s - d_m/2)$ securing the proper periodicity. Then the replacements (25) lead to the form

$$r_m^m(s) = \frac{1}{(m-1)!} \sum_{\mathbf{r}}^{m-1} C_{m-1}^{\mathbf{r}} \prod_{i=1}^{m-1} d_i^{r_i-1} {}^iB_{r_i} \Psi_{d_m}(s - d_m/2). \quad (37)$$

Using the explicit form of the operator ${}^i\mathbf{B}_{r_i}$ we arrive at

$$r_m^m(s) = \frac{1}{(m-1)!} \sum_{\mathbf{r}} C_{m-1}^{\mathbf{r}} \prod_{i=1}^{m-1} \tau_i^{r_i-1} \sum_{p_i=0}^{\tau_i/d_i-1} B_{r_i} \left(1 - \frac{\lambda_{p_i} d_i}{\tau_i}\right) \Psi_{d_m} \left(s - \sum_{i=1}^m \lambda_{p_i} d_i\right). \quad (38)$$

It should be underlined here that the values of "periods" τ_i depends on the order of the elements in the set \mathbf{d}^m , and in the above formula we have $\tau_1 = \text{LCM}(d_m, d_1)$, $\tau_2 = \text{LCM}(d_m, d_1, d_2)$, \dots , $\tau_{m-1} = \text{LCM}(\mathbf{d}^m)$, $\tau_m = d_m$, so that $p_m \equiv 0$. The last formula can be rewritten in the symbolic form

$$r_m^m(s) = \frac{1}{(m-1)!} \prod_{i=1}^{m-1} \tau_i^{-1} \left[\sum_{i=1}^{m-1} \tau_i \left\{ \sum_{p_i=0}^{\tau_i/d_i-1} {}^iB \left(1 - \frac{\lambda_{p_i} d_i}{\tau_i}\right) \right\} \right]^{m-1} \Psi_{d_m} \left(s - \sum_{i=1}^m \lambda_{p_i} d_i\right). \quad (39)$$

Hence, combination of the last expression with the recursion (28) provides a new procedure for calculation of the restricted partition function $V(s, \mathbf{d}^m)$.

4 Explicit formula for restricted partition function

The same approach can be useful in order to produce a formula for $V(s, \mathbf{d}^m)$, starting from its polynomial part (13) which can be written as

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!} \left(\prod_{i=1}^m d_i^{-1} \right) \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \left(\sum_{i=1}^m d_i {}^iB(1/2) \right)^l. \quad (40)$$

It is clear that there exist several equivalent forms of partition function (like two possible forms of $R_1^2(s)$ discussed above), and symmetry considerations can help in selection of the most symmetric form. In the discussed case the form is chosen by a selection of the factor $1/d_i$ for the replacement $1/d_i \Rightarrow \Psi_{d_i}(s - d_i/2)$ and application of (25) to the result.

In general situation we have m choices of $1/d_i$ to start with, and because the result doesn't depend of such choice, it is natural to seek an expression symmetrized w.r.t. the starting element d_i . The latter problem is equivalent to the problem of presentation of the symmetric polynomial

$$\left(\sum_{i=1}^m d_i \right)^l = \sum_{l=0}^l C_l^{\mathbf{r}} \prod_{i=1}^m d_i^{r_i}$$

as a sum of m symmetric polynomials, each of them missing only one of terms d_i . Introducing a function $z(\mathbf{r})$ counting number of zero components of the vector \mathbf{r} , one can write the above expression as

$$\sum_{i=1}^m \sum_{\mathbf{r}} \frac{1}{z(\mathbf{r})} C_l^{\mathbf{r}} \prod_{\substack{n=1 \\ n \neq i}}^m d_n^{r_n}. \quad (41)$$

Using this result we have

$$\left(\prod_{i=1}^m d_i^{-1} \right) \left(\sum_{i=1}^m d_i {}^iB(1/2) \right)^l = \sum_{i=1}^m \frac{1}{d_i} \sum_{\mathbf{r}} \frac{1}{z(\mathbf{r})} C_l^{\mathbf{r}} \prod_{\substack{n=1 \\ n \neq i}}^m d_n^{r_n-1} B_{r_n}(1/2). \quad (42)$$

The polynomial part (40) suitable for conversion into $V(s, \mathbf{d}^m)$ has the form

$$V_1(s, \mathbf{d}^m) = \frac{1}{(m-1)!} \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \sum_{i=1}^m \frac{1}{d_i} \sum_{\mathbf{r}} \frac{1}{z(\mathbf{r})} C_l^{\mathbf{r}} \prod_{\substack{n=1 \\ n \neq i}}^m d_n^{r_n-1} B_{r_n}(1/2). \quad (43)$$

The explicit expression for the restricted partition function reads

$$V(s, \mathbf{d}^m) = \frac{1}{(m-1)!} \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \sum_{i=1}^m \left[\sum_{\mathbf{r}} \frac{1}{z(\mathbf{r})} C_l^{\mathbf{r}} \prod_{\substack{n=1 \\ n \neq i}}^m d_n^{r_n-1} {}^n \mathbf{B}_{r_n} \right] \Psi_{d_i}(s - d_i/2). \quad (44)$$

Using the actual form of the operators ${}^n \mathbf{B}_{r_n}$ (25) it is rewritten as

$$V(s, \mathbf{d}^m) = \frac{1}{(m-1)!} \sum_{l=0}^{m-1} C_{m-1}^l s^{m-1-l} \sum_{i=1}^m \left[\sum_{\mathbf{r}} \frac{1}{z(\mathbf{r})} C_l^{\mathbf{r}} \prod_{\substack{n=1 \\ n \neq i}}^m \tau_{n,i}^{r_n-1} \sum_{p_n=0}^{\tau_{n,i}/d_n-1} B_{r_n} \left(1 - \frac{\lambda_{p_n} d_n}{\tau_{n,i}}\right) \right] \Psi_{d_i} \left(s - \sum_{n=1}^m \lambda_{p_n} d_n \right). \quad (45)$$

It should be noted here that the values of "periods" τ depends on selected value d_i what is reflected by an additional subscript i :

$$\begin{array}{ccccccc} \tau_{1,1} = d_1 & \tau_{2,1} = \text{LCM}(d_1, d_2) & \tau_{3,1} = \text{LCM}(d_1, d_2, d_3) & \dots & \tau_{m,1} = \text{LCM}(\mathbf{d}^m) \\ \tau_{1,2} = \text{LCM}(d_1, d_2) & \tau_{2,2} = d_2 & \tau_{3,2} = \text{LCM}(d_1, d_2, d_3) & \dots & \tau_{m,2} = \text{LCM}(\mathbf{d}^m) \\ \tau_{1,3} = \text{LCM}(d_1, d_3) & \tau_{2,3} = \text{LCM}(d_1, d_2, d_3) & \tau_{3,3} = d_3 & \dots & \tau_{m,3} = \text{LCM}(\mathbf{d}^m) \\ \dots & \dots & \dots & \dots & \dots \\ \tau_{1,m} = \text{LCM}(d_1, d_m) & \tau_{2,m} = \text{LCM}(d_1, d_2, d_m) & \tau_{3,m} = \text{LCM}(d_1, d_2, d_3, d_m) & \dots & \tau_{m,m} = d_m \end{array}$$

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