Math 108 Final Exam Solutions, Winter 2005

Problem 1 *(9 points)* Consider the sentence, "Everyone has a friend, but not everyone has an enemy." (Universe = set of all people)

a. Express that sentence symbolically.

$(\forall x)(\exists y)(y \text{ is a friend of } x) \wedge \sim(\forall x)(\exists y)(y \text{ is an enemy of } x)$

b. Write a symbolic denial of your answer to part a.

$\sim[(\forall x)(\exists y)(y \text{ is a friend of } x) \wedge \sim(\forall x)(\exists y)(y \text{ is an enemy of } x)]$
$(\exists x)(\forall y)(y \text{ is not a friend of } x) \vee (\forall x)(\exists y)(y \text{ is an enemy of } x)$

c. Translate your answer to part b. into English.

Someone is friendless, or everyone has enemies.

Problem 2 *(10 points)* Let $a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n > 2$. Use PCI or WOP to prove that $a_n = n$ for all $n \in \mathbf{N}$. Proof by PCI. We can see that the claim is true for $n = 1$ and $n = 2$. Suppose n is a natural number greater than 2, such that $a_m = m$ for all $m < n$. Then

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(n-1) - (n-2) \\ &= 2n - 2 - n + 2 \\ a_n &= n \end{aligned}$$

That is, if $a_m = m$ for all $m < n$, then $a_n = n$; thus, by PCI, the claim is true for all $n \in \mathbf{N}$.

Problem 3 *(10 points)* Recall that $p \equiv_7 q$ iff $p - q = 7k$ for some integer k. Prove that $\equiv_7$ is an equivalence relation on $\mathbf{Z}$.
$\equiv_7$ is reflexive: Let $x \in \mathbf{Z}$. Then $x - x = 0 = 0 \cdot 7$, so $x \equiv_7 x$.
$\equiv_7$ is symmetric: Suppose $x \equiv_7 y$. Then $x - y = 7k$ for some $k \in \mathbf{Z}$. Thus, $y - x = 7(-k)$, so $y \equiv_7 x$.
$\equiv_7$ is transitive: Suppose $x \equiv_7 y$ and $y \equiv_7 z$. Then $x - y = 7k$ for some $k \in \mathbf{Z}$. Also, $y - z = 7l$ for some $l \in \mathbf{Z}$. Thus, $x - z = (x - y) + (y - z) = 7k + 7l = 7(k + l)$, so $x \equiv_7 z$.

Problem 4 *(10 points)* True or false:

[T]/F Ø is finite.

T/[F] There are uncountably many rational numbers.

T/[F] Every relation is a function.

T/[F] If $S = \{1, 2, 3\}$, then $S \subseteq \mathcal{P}(S)$.

$\boxed{\text{T}}$/F The number 0 is even.

Problem 5 *(10 points)* Prove that for any real number x: if x^2 is irrational, then x is irrational. Proof by contraposition: suppose x is rational. Then for some integers p and q (with $q \neq 0$), $x = \frac{p}{q}$. Thus $x^2 = \frac{p^2}{q^2}$. Since p^2 and q^2 are integers (with $q^2 \neq 0$), it follows that x^2 is rational. Now, since "x is rational" implies "x^2 is rational", we conclude the contrapositive of this claim, namely: If x^2 is irrational, then x is irrational.

Problem 6 *(8 points)* If possible, give an example of:

a. A bijection from $\mathbf{N} \cup \{\pi\}$ to $\mathbf{N}$

$$f(x) = \begin{cases} 1 & \text{if } x = \pi \\ x+1 & \text{if } x \in \mathbf{N} \end{cases}$$

b. An injection from $\mathbf{N}$ to (0,1)

$g(n) = \frac{1}{n+1}$

c. A surjection from $\mathbf{Z}$ to $\mathbf{R}$

Not possible.

d. A set S such that $\overline{\overline{\mathbf{R}}} < \overline{\overline{S}}$.

$S = \mathcal{P}(\mathbf{R})$

Problem 7 *(10 points)* For $n \in \mathbf{N}$, let $A_n = [1 + \frac{1}{n}, 2 + n]$. Find:

a.

$$\bigcap_{n=1}^{\infty} A_n$$

The intersection is $A_1 = [2, 3]$.

b.

$$\bigcup_{n=1}^{\infty} A_n$$

The union is $(1, \infty)$.

Problem 8 *(10 points)* Suppose $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $g \circ f : A \xrightarrow{1-1} C$. Prove that f is one to one. Proof: Suppose that, for some x, $y \in A$, $f(x) = f(y)$. Since g is a function (or, "applying g to both sides"), we find that $g(f(x)) = g(f(y))$. That is, $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is 1-1, this implies that $x = y$. So, f is 1-1.

Problem 9 *(10 points)* Let $A = (-\infty, -10) \cup (-1, \pi] \cup \mathbf{N}$. Prove that $\overline{\overline{A}} = \mathbf{c}$.
Note that $(0,1) \subseteq A \subseteq \mathbf{R}$, so $\mathbf{c} \leq \overline{\overline{A}} \leq \mathbf{c}$. By Cantor-Schröder-Bernstein, $\overline{\overline{A}} = \mathbf{c}$.

Problem 10 *(10 points)* Use PMI to prove that for any natural number n, $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$.
For $n = 1$, the claim says that $1 + \frac{1}{2} = 2 - \frac{1}{2}$, which is true.
Suppose that the claim holds for some natural number n; that is, suppose $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$. It follows that

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}$$

. That is, if the claim holds for n, it also holds for $n + 1$. By PMI, it is true for all $n \in N$.

Extra Credit
Recall that $\mathbf{Z}_7 = \mathbf{Z}/_{\equiv_7}$, which is the partition of $\mathbf{Z}$ arising from the equivalence relation in problem 3. Define the function $f : \mathbf{Z}_7 \longrightarrow \mathbf{Z}_7$ by $f(\overline{x}) = \overline{x^2}$. Which of the following belong to Rng(f)?
$\overline{0}$ $\overline{1}$ $\overline{2}$ $\overline{3}$

I was going to turn this into a sermon on quadratic residues, but I won't. Instead, let's just compute them.

$$\begin{array}{lllllll} f(\overline{0}) & = & \overline{0^2} & = & \overline{0} & & \\ f(\overline{1}) & = & \overline{1^2} & = & \overline{1} & & \\ f(\overline{2}) & = & \overline{2^2} & = & \overline{4} & & \\ f(\overline{3}) & = & \overline{3^2} & = & \overline{9} & = & \overline{2} \\ f(\overline{4}) & = & \overline{4^2} & = & \overline{16} & = & \overline{2} \\ f(\overline{5}) & = & \overline{5^2} & = & \overline{25} & = & \overline{4} \\ f(\overline{6}) & = & \overline{6^2} & = & \overline{36} & = & \overline{1} \end{array}$$

So, the range of f is $\{\overline{0}, \overline{1}, \overline{2}, \overline{4}\}$.

Here's the question I was *going* to put on as extra credit before I decided it was over the top. I'll post the solution later...you should at least have a few hours alone with the problem, sans chatter.

Extra Credit
A number is called *algebraic* if it is the root of some polynomial with integer coefficients. For instance, $x = \frac{3}{4}$ is algebraic, because it satisfies the equation $4x - 3 = 0$. (Similarly, every rational number is algebraic.) Also, $x = \sqrt{2}$ is algebraic because it satisfies the equation $x^2 - 2 = 0$. Prove that there are countably many algebraic numbers.