Problem 1 *(16 points)* Let $M(x, y)$ be the claim that x is y's mother, and let $F(x, y)$ be the claim that x is y's father. Translate the following symbolic sentences into natural-sounding English. (Which means: don't say "there exists a person who is my uncle" when you could say "I have an uncle".) The universe is the set of all people.

a. $(\forall a)(\exists! b)M(b, a)$

Each person has exactly one mother.

b. $\sim(\forall a)(\exists b)F(a, b)$

Not everyone is a father.

Translate these from English into symbolic form, using only the predicates M and F; for instance, do not create a new predicate for "x is y's grandfather".

c. Vader is Luke's father only if Amidala is Leia's mother.

$M(\text{Amidala}, \text{Leia}) \Rightarrow F(\text{Vader}, \text{Luke})$

d. Leroy is Brad's grandfather.

$(\exists x)(F(\text{Leroy}, x) \wedge (F(x, \text{Brad}) \vee M(x, \text{Brad})))$

Problem 2 *(12 points)* Let A be the set of odd integers, let B be the set of positive integers, let $C = (-4, 5)$, and let $D = \{1, \{2, 3\}, \emptyset\}$.

a. Find $A - B$.

$A-B$ is the set of negative odd numbers $= \{-1,-3,-5,\ldots\} = \{n : n = 1-2k$ for some $k \in \mathbb{N}\}$

b. Find $B \cap C$.

$B \cap C = \{1,2,3,4\}$

c. Find $\mathcal{P}(D)$.

$\mathcal{P}(D) = \{\emptyset, \{1\}, \{\{2,3\}\}, \{\emptyset\}, \{1, \{2,3\}\}, \{1, \emptyset\}, \{\{2,3\}, \emptyset\}, \{1, \{2,3\}, \emptyset\}\}$

Problem 3 *(15 points)* Prove the following claim: If x is irrational and $xy = 1$, then y is irrational. (Universe = $\mathbf{R}$)

Proof by Contradiction. Suppose x is irrational, $xy=1$, and y is rational. Then there exist integers p and q, $q \neq 0$, such that $y = p/q$. Therefore $x\left(\frac{p}{q}\right) = 1$, so $xp = q$. Since $q \neq 0$, it follows that $p \neq 0$, so $x = \frac{q}{p}$. Thus x is rational, contradicting our assumption that x is irrational.

Therefore, if x is irrational and $xy = 1$, then y is irrational.

Problem 4 *(5 points)* Distribute the negation as far as possible through the following statement.

$\sim(\exists m)[P(m) \wedge (\forall n)(Q(m,n) \Rightarrow R(n))]$ iff $(\forall m)\sim[P(m) \wedge (\forall n)(Q(m,n) \Rightarrow R(n))]$

iff $(\forall m)[\sim P(m) \vee \sim(\forall n)(Q(m,n) \Rightarrow R(n))]$

iff $(\forall m)[\sim P(m) \vee (\exists n)\sim(Q(m,n) \Rightarrow R(n))]$

iff $\boxed{(\forall m)[\sim P(m) \vee (\exists n)(Q(m,n) \wedge \sim R(n))]}$

Problem 5 *(10 points)* Let $A_n = [2 + \frac{1}{n}, 3 + \frac{1}{n}]$ for all natural numbers n. Find:

a.

$$\bigcap_{n=1}^{\infty} A_n = \{3\}$$

b.

$$\bigcup_{n=1}^{\infty} A_n = (2, 4]$$

Problem 6 *(15 points)* Let A, B, and C be sets. Prove that if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

Suppose $A \subseteq B \cup C$ and $A \cap B = \emptyset$. To show $A \subseteq C$, we let x be an element of A. Since $A \subseteq B \cup C$, it follows that $x \in B \cup C$. Hence $x \in B$ or $x \in C$. But, if $x \in B$, then $x \in A \cap B$; this contradicts our assumption that $A \cap B = \emptyset$. Therefore $x \notin B$ and, consequently, $x \in C$. That is, if $x \in A$ then $x \in C$.

Thus $A \subseteq C$, as we were to prove.

Problem 7 *(15 points)* Use the Principle of Mathematical Induction to prove that $3 + 7 + 11 + \ldots + (4n - 1) = 2n^2 + n$.

Let $P(n)$ be the above claim.

Base case: Since $3 = 2(1)^2 + 1$, $P(1)$ is true.

Inductive step: Suppose $P(n)$ holds, for some n. Then:

$$3 + 7 + 11 + \cdots + (4n-1) + (4(n+1)-1) = (2n^2+n) + (4(n+1)-1)$$
$$= 2n^2 + n + 4n + 3$$
$$= 2n^2 + 5n + 3$$
$$= 2n^2 + 4n + 2 + n + 1$$
$$= 2(n+1)^2 + n + 1.$$

Thus, $P(n) \Rightarrow P(n+1)$, so by PMI, $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 8 *(12 points)* Find a counterexample to each of the following claims, if one exists. If there is no counterexample, just say so; no proof is required.

a. For all sets A, B, and C: If $B \subseteq C$, then $A - B \subseteq A - C$.

$A = \{1\}$, $B = \emptyset$, $C = \{1\}$.

b. If $\mathcal{A}$ is a family of sets, then

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A.$$

$\mathcal{A} = \emptyset$.

c. If $a|b$ and $c|d$, then $(a+c) \mid (b+d)$.

$a = b = d = 2$, $c = 1$.

Problem 9 *(10 extra credit points)* Recall that the *symmetric difference* of two sets A and B is given by $A \Delta B = (A - B) \cup (B - A)$. Prove or disprove: For any two sets A and B, $\mathcal{P}(A \Delta B) = \mathcal{P}(A) \Delta \mathcal{P}(B)$.

Note that for any set S, $\emptyset \subseteq S$, so $\emptyset \in \mathcal{P}(S)$.

In particular: $\emptyset \in \mathcal{P}(A)$, $\emptyset \in \mathcal{P}(B)$, $\emptyset \in \mathcal{P}(A \Delta B)$.

But since $\emptyset \in \mathcal{P}(A)$ and $\emptyset \in \mathcal{P}(B)$, we have

$\emptyset \notin \mathcal{P}(A) \Delta \mathcal{P}(B)$.

Thus $\emptyset \in \mathcal{P}(A \Delta B)$ but $\emptyset \notin \mathcal{P}(A) \Delta \mathcal{P}(B)$, so

$$\mathcal{P}(A \Delta B) \neq \mathcal{P}(A) \Delta \mathcal{P}(B)$$