Problem 1 *(18 points)*

1. Let R be the only equivalence relation on $A = \{1, 2, 3, 4, 5\}$ that contains (1,5), (4,3), and (2,3), but not (4,5). Also, let S be the equivalence relation on A given by xSy iff $2|x - y$. Let $T = R \cap S$. Draw digraphs for R, S, and T.

   Solution: I'll describe them. In every digraph, there should be loops at every number (reflexivity). In the digraph for R, 1 and 5 should be connected by a pair of edges; also, 2, 3, and 4 should all be connected (6 edges between them in all). In the digraph for S, 1, 3, and 5 should be connected (6 edges between them), and 2 and 4 should be connected by a pair of edges. In the digraph for T, a pair of edges should connect 1 to 5 and another pair should connect 2 to 4. The only edges in the T digraph are those edges that appeared in both R and S. Since 3 is not connected to 1 or 5 in R and 3 is not connected to 2 or 4 in S, 3 should be completely isolated in T.

2. What is A/R? $\{\{1, 5\}, \{2, 3, 4\}\}$
3. What is A/S? $\{\{1, 3, 5\}, \{2, 4\}\}$
4. What is A/T? $\{\{1, 5\}, \{2, 4\}, \{3\}\}$

Problem 2 *(16 points)*

1. Which one of the following statements is always true?
   - $\text{Dom}(g \circ f) \subseteq \text{Dom}(f)$ Yes
   - $\text{Dom}(f) \subseteq \text{Dom}(g \circ f)$ No
2. Which one of the following statements is always true?
   - $\text{Rng}(g \circ f) \subseteq \text{Rng}(g)$ Yes
   - $\text{Rng}(g) \subseteq \text{Rng}(g \circ f)$ No
3. Pick one of the unreliable statements above, and find a counterexample.

Consider $f : (-\infty, 1) \to (-\infty, 1)$ by $f(x) = x$ and $g : (-1, \infty) \to (-1, \infty)$ by $g(x) = x$. This is the long way of saying $f = \text{Id}_{(-\infty,1)}$ and $g = \text{Id}_{(-1,\infty)}$. The composition of these two functions is $g \circ f = \text{Id}_{(-1,1)}$. We have $\text{Dom}(f) = (-\infty, 1)$, $\text{Rng}(g) = (-1, \infty)$, and $\text{Dom}(g \circ f) = \text{Rng}(g \circ f) = (-1, 1)$. This is a counterexample to both of the unreliable statements indicated above.

Problem 3 *(15 points)* Let A and B be nonempty sets, let $f : A \longrightarrow B$, and define the relation F on A by: xFy iff $f(x) = f(y)$. Is F an equivalence relation? Prove your answer.

Solution: Yes, it is an equivalence relation:

1. F is reflexive. For any $x \in A$, $f(x) = f(x)$, so xFx.
2. F is symmetric. For any $x, y \in A$: if xFy, then $f(x) = f(y)$. Thus, $f(y) = f(x)$, so yFx.
3. F is transitive. Let x, y, and z be elements of A such that xFy and yFz. Then $f(x) = f(y)$ and $f(y) = f(z)$, so it follows that $f(x) = f(z)$; that is, xFz.

And now, going a little past the problem: the equivalence classes for this relation are the preimages of singletons in B. That is, if $x \in A$ and $f(x) = b \in B$, then the equivalence class x/F is described by:

$$\begin{aligned} x/F &= \{y \in A : xFy\} \\ &= \{y \in A : f(x) = f(y)\} \\ &= \{y \in A : f(y) = b\} \\ &= \{y \in A : f(y) \in \{b\}\} \\ &= f^{-1}(\{b\}) \end{aligned}$$

Problem 4 *(18 points)* Recall that the factorial numbers are determined by this inductive definition: $1! = 1$ and, for $n > 1$, $n! = n \cdot (n-1)!$. Prove that for all natural numbers n, $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$.

Proof by PMI. I'd like to start by pointing out: we aren't trying to prove that the factorials obey their inductive definition—that is given.
If $P(n)$ says that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$, then we are trying to prove that $P(n)$ is true for all $n \in \mathbf{N}$.
Base Case, $n = 1$: $1 \cdot 1! = 1 \cdot 1 = 1$. Also, $(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$, so $1 \cdot 1! = (1+1)! - 1$. Thus $P(1)$ is true.
Now suppose $P(n)$ is true; that is, suppose $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$. (This is our inductive hypothesis.) It follows that:

$$\begin{aligned}
1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n! + (n+1)\cdot(n+1)! &= (n+1)! - 1 + (n+1)\cdot(n+1)! \\
&= (n+1)!(1+n+1) - 1 \\
&= (n+1)!(n+2) - 1 \\
&= (n+2)! - 1 \\
1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (n+1)\cdot(n+1)! &= ((n+1)+1)! - 1
\end{aligned}$$

That is, $P(n) \Rightarrow P(n+1)$. By the PMI, $P(n)$ is true for all $n \in \mathbf{N}$.

Problem 5 *(18 points)* Let $a_1 = 3$ and $a_2 = 9$, and for all $n > 2$, let $a_n = 2a_{n-1} + 3a_{n-2}$. Use either PCI or WOP to prove that $a_n = 3^n$ for all $n \in \mathbf{N}$.

Proof by PCI: Let $S = \{n \in \mathbf{N} : a_n = 3^n\}$. We intend to show that $S = \mathbf{N}$. Suppose that for some $n \in \mathbf{N}$, every natural number less than n belongs to S. (This is our inductive hypothesis.) We must show that $n \in S$, too. If $n = 1$, then $n \in S$ because $a_1 = 3 = 3^1$. If $n = 2$, then $n \in S$ because $a_2 = 9 = 3^2$. Finally, if $n > 2$, then $n-1$ and $n-2$ are natural numbers less than n; thus, by our inductive hypothesis, they belong to S. That is, $a_{n-2} = 3^{n-2}$ and $a_{n-1} = 3^{n-1}$. Thus,

$$\begin{aligned}
a_n &= 2a_{n-1} + 3a_{n-2} \\
&= 2\cdot 3^{n-1} + 3\cdot 3^{n-2} \\
&= 2\cdot 3^{n-1} + 3^{n-1} \\
&= 3^{n-1}(2+1) \\
&= 3\cdot 3^{n-1} \\
a_n &= 3^n
\end{aligned}$$

In other words, whenever S contains all predecessors of n, it also contains n. By the PCI, $S = \mathbf{N}$ and the claim is true for all $n \in \mathbf{N}$.

Proof by WOP. Overview: a typical WOP proof considers the "set of counterexamples" to the claim. We assume the claim is false, so this set is nonempty; by WOP, it has a smallest element. Something about this element produces a contradiction, implying that the claim is true after all. Compare this overview to the following proof.

Let $T = \{n \in \mathbf{N} : a_n \neq 3^n\}$. (Referring to the set S from the previous proof, $T = \mathbf{N} - S$.) Suppose there is a natural number n such that $a_n \neq 3^n$. Then T is nonempty. By the WOP, it follows that T contains a smallest element, which we may call m. Note that $m \neq 1$ because $a_1 = 3 = 3^1$. Also, $m \neq 2$ because $a_2 = 9 = 3^2$. Thus, $m > 2$. Now $m-1$ and $m-2$ are smaller than m, which implies that they do not belong to T. Now they are natural numbers not in T, so they must fail the condition "$a_n \neq 3^n$"; that is, $a_{m-1} = 3^{m-1}$ and $a_{m-2} = 3^{m-2}$. But then:

$$\begin{aligned}
a_m &= 2a_{m-1} + 3a_{m-2} \\
&= 2 \cdot 3^{m-1} + 3 \cdot 3^{m-2} \\
&= 2 \cdot 3^{m-1} + 3^{m-1} \\
&= 3^{m-1}(2+1) \\
&= 3 \cdot 3^{m-1} \\
a_m &= 3^m
\end{aligned}$$

This means that $m \notin T$ after all, a contradiction. Thus $T = \emptyset$; there exists no natural number n such that $a_n \neq 3^n$. In other words, $a_n = 3^n$ for all natural numbers n.

Problem 6 *(15 points)* Suppose $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $g \circ f : A \xrightarrow{1-1} C$. Prove that f is one to one.

Proof. Suppose x and y are elements of A, such that $f(x) = f(y)$. Applying g to both sides, we find that $g(f(x)) = g(f(y))$, which means $(g \circ f)(x) = (g \circ f)(y)$. But $g \circ f$ is 1-1, so $x = y$. Since $f(x) = f(y) \Rightarrow x = y$, we conclude that f is 1-1.