Math 108 Ballinger

Final Exam Review Packet

This packet is intended to touch on the main points of the class and give you examples of different ways concepts could be tested.

Chapter 1

Know what is and what isn't a proposition; know how to change and combine propositions with such logical connectives as conjunction (and), disjunction (or), negation (not), implication (if/then), and biconditional (iff). I'm especially interested in your ability to translate statements from English to symbolic form and vice-versa. I would like you to understand why, in the two translations below, one statement is written with implication and the other with conjunction: (the universe is the set of all animals)

"Some cats are tigers"$\longrightarrow$"$(\exists x)(x$ is a cat $\wedge\, x$ is a tiger$)$"

"All tigers are cats"$\longrightarrow$"$(\forall x)(x$ is a tiger $\Rightarrow x$ is a cat$)$"

Sample problem: Let A be the set of people, B the set of continents on Earth, and C the set of cities on Earth. Let $h(x, y)$ be the statement "x has been to y" (as in, Albert has been to Prague); let $f(y, z)$ be the statement "y is on z" (as in, Davis is on North America). Express these sentences symbolically:

1. Someone has been to two continents.
2. Nobody has been to a city on every continent.

A common error I've seen on your midterms, especially in proofs by induction, is: "$n \Rightarrow n+1$". If n is a number, it has no truth value; it cannot be true or false, but it might be 2 or 7. Thus it cannot be an antecedent or a consequent. One correct way to express that idea that Case n implies Case $n+1$ would be "$P(n) \Rightarrow P(n+1)$", where $P(n)$ is a certain proposition about n (such as, "n has a prime factor" or whatever). In general, a whole equation (like "$a_n = 3^n$") can sit on either side of an implication or other logical connective, but a solitary expression (like "a_n" or "12") cannot.

The implication $A \Rightarrow B$ is equivalent to its *contrapositive*, $\sim B \Rightarrow \sim A$, but not equivalent to its *converse*, $B \Rightarrow A$, or its *inverse*, $\sim A \Rightarrow \sim B$. This is why "Proof by contrapositive" is a valid and popular proof technique, but you've never heard of "Proof by converse". (The converse, by the way, is equivalent to the inverse.) It is unlikely that I will ask you questions specifically about these relationships. It is more likely that a misunderstanding here (if you have one) will affect your proofs.

From time to time I've said something like, "Let S be the truth set of the claim". For example, the claim (about integers) that $n = n^2$ has truth set $\{0, 1\}$, while the claim (about natural numbers) that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ has truth set $\mathbb{N}$. I won't specifically ask you what "truth set" means, but understanding this concept may help you with proofs by PMI and PCI.

For the sakes of you, me, and future generations, please find the definition of Divides on p.30, and read that paragraph. Please decide whether zero is even, odd, neither, or both (using the definitions in that paragraph), and be very sure of your answer. How sure? Yes, go ahead and prove it.

Basic Proof Methods are, of course, key. Direct, Contraposition, Contradiction. Proof by exhaustion (cases) is hidden somewhere in 1.4. I want to emphasize that a proof may involve more than one of these. You may have several cases, some of which you prove directly, and some of which you prove by contraposition. During a recent proof by PMI, I found it useful to prove the inductive step by contradiction. Use whatever tools you have to get the job done. As a rule of thumb, if the theorem I'm trying to prove gives me something *positive* to work with, I try a direct proof first. For example:

"Prove that if x and y are rational, then $x + y$ is rational." In a direct proof, I would assume that x and y are rational, which means they can be expressed as $x = \frac{a}{b}$ and $y = \frac{c}{d}$. This gives me something to work with.

Here's the opposite kind of example:

"Prove that if x is irrational, then $2x$ is irrational." Now in a direct proof I could assume that x is irrational, which means it cannot be expressed in a certain form. This is, in some sense, *negative* information: irrational numbers *don't* obey some rule. That's nearly useless. Notice that the consequent we're trying to prove is similarly negative. In this case, I would go to a proof by Contraposition, which flips things around and turns negative information into positive information. The contrapositive of the given claim is: "If $2x$ is rational, then x is rational". Suddenly, I get to assume that $2x = \frac{p}{q}$ for some integers p and q, and the proof is halfway done. Contraposition is a little finicky, though; it requires just the right kind of implication.

Contradiction is more robust. Here's a claim: "If x is rational and y is irrational, then $x + y$ is irrational". We've got 2 variables on the loose. In a direct proof, I could assume that x is rational and y isn't. That gives me solid, positive information about x but not y. That turns out to not be enough information. In a proof by contraposition, I could assume that $x + y$ is rational—which is nice, but it's still only one piece of solid info. To get the best of both worlds, I would turn to my old friend, Contradiction. I get to assume x is rational, y is irrational, and $x + y$ is rational. Here I see 2 pieces of positive information and one piece of negative. We focus on the positive for now, and see what happens: there exist integers a, b, c, and d such that $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$, where b and d are nonzero. I know $y = (x + y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$, and suddenly y is rational. This contradicts the piece of negative info we hadn't used yet, and the proof is done.

OK, I'm talking too much. Let me get back to brief mode. I want you to know what's going on in the blue box on p.45, but not because you've memorized it—I want you to see that this is a special case of the box on p.38. Think about the connection between the box on p.48 and a typical proof that some function is 1-1.

Oh, one of my pet peeves is at the bottom of p.54. A few of you used that proof technique on the last midterm: you assumed the claim was true for $n + 1$ and proved something like $k = k$. This is NOT a valid proof technique.

Well, this is all I have so far, as of 2:20am Monday morning. I'll add to this document this afternoon.