Math 108 Practice Final

Problem 1 **a.** State the definition of "$3|n$".

b. Use this definition to prove that the sum of any three consecutive integers is divisible by 3.

c. Is the sum of any four consecutive integers divisible by 4?

Problem 2 A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, or *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

a. Prove that if f and g are odd, then fg is even.

b. Prove that if f and g are odd, then $f \circ g$ is odd.

Problem 3 Prove that for sets A and B, $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Problem 4 If possible, find sets A and B and a function $f : A \to B$ such that:

a. $\aleph_0 \leq \overline{\overline{A}} = \overline{\overline{B}}$ and f is onto, but not 1-1

b. $\overline{\overline{A}} = \overline{\overline{B}} < \aleph_0$ and f is 1-1, but not onto

c. A is uncountable and B is finite

d. A is denumerable, B is uncountable, and f is onto.

Problem 5 For $n \in \mathbb{N}$, let $A_n = [1 - \frac{1}{n}, 2 - \frac{1}{n}] \cup \{n\}$. Find:

a.

$$\bigcap_{n=1}^{\infty} A_n$$

b.

$$\bigcup_{n=1}^{\infty} A_n$$

c. The symmetric difference $A_1 \Delta A_3$

Problem 6 Let $\mathcal{A}$ be a partition of A. Define the relation Q on A by: xQy iff for some $S \in \mathcal{A}$, $x \in S$ and $y \in S$. Prove that Q is reflexive.

Problem 7 Prove or disprove:

a. For all sets A, B, and C: If $A - B \subseteq C$, then $C - B \subseteq A$.

b. For all sets A, B, and C: If $A - B \subseteq C$, then $A \subseteq B \cup C$.

Problem 8 Let S be a fixed subset of $\mathbf{R}$. Define the relation $\star$ on $\mathcal{P}(\mathbf{R})$ by $A \star B$ iff $A - S = B - S$.

a. Prove that $\star$ is an equivalence relation.

b. ♣ If $S = \{0, 1\}$ and $A = [0, 1]$, describe $A/\star$.

Problem 9 Let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3 - x$. Find:

a. $f([-1, 1])$

b. $f^{-1}(\{0\})$

c. A set S such that $f|_S$ is 1-1 and onto $\mathbf{R}$

Problem 10 Use the Principle of Mathematical Induction to prove the following formula for the sum of a geometric series:

$$\sum_{k=0}^{n} a \cdot r^k = a\left(\frac{1 - r^{n+1}}{1 - r}\right)$$

Problem 11 ♣ Use the WOP to prove that every natural number is interesting.

Problem 12 Let $a_1 = 1$, $a_2 = 4$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n > 2$. Use PCI or WOP to prove that, for every natural number n, $a_n = n \cdot 2^{n-1}$.

Problem 13 Prove that the set $[1, 2) \cup (3, 4)$ has cardinal number $\mathbf{c}$.

Problem 14 Repeat problem 13, without using Cantor-Schröder-Bernstein. (That is, construct a bijection.)

Note: Problems marked with the club (♣) are somewhat wacky. If they don't make sense to you, don't despair.