Math 108 Practice Final Solutions

Problem 1 **a.** State the definition of "$3|n$".
For any integer n, $3|n$ if and only if there exists an integer k such that $n = 3k$.

b. Use this definition to prove that the sum of any three consecutive integers is divisible by 3.

Let n be the first of the three; then the others are $n+1$ and $n+2$. Their sum is $n+(n+1)+(n+2) = 3n+3 = 3(n+1)$, so there exists an integer k (specifically, $k = n+1$) such that $n+(n+1)+(n+2) = 3k$.

c. Is the sum of any four consecutive integers divisible by 4?

No. $-1+0+1+2 = 2$. In fact, the sum of four consecutive integers is *never* divisible by 4: $n+(n+1)+(n+2)+(n+3) = 4n+6 = 4n+4+2 = 4(n+1)+2$, so the sum of four consecutive integers is congruent to 2 modulo 4.

Problem 2 A function $f : \mathbf{R} \to \mathbf{R}$ is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$, or *even* if $f(-x) = f(x)$ for all $x \in \mathbf{R}$.

a. Prove that if f and g are odd, then fg is even.

Suppose f and g are odd. It follows that for any $x \in \mathbf{R}$, $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$. Since $(fg)(-x) = (fg)(x)$ for all x, fg is even.

b. Prove that if f and g are odd, then $f \circ g$ is odd.

Suppose f and g are odd. It follows that for any $x \in \mathbf{R}$, $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$. Since $(f \circ g)(-x) = -(f \circ g)(x)$ for all $x \in \mathbf{R}$, $(f \circ g)$ is odd.

Problem 3 Prove that for sets A and B, $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Part 1: $(A \subseteq B) \Rightarrow (\mathcal{P}(A) \subseteq \mathcal{P}(B))$.
Suppose $A \subseteq B$; to prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we let $X \in \mathcal{P}(A)$. It follows that $X \subseteq A$, by the definition of power set. Since $X \subseteq A$ and $A \subseteq B$, we have $X \subseteq B$; thus, $X \in \mathcal{P}(B)$. Thus, if $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Part 2: $(\mathcal{P}(A) \subseteq \mathcal{P}(B)) \Rightarrow (A \subseteq B)$.
Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We will prove that $A \subseteq B$, but without chasing elements this time. Instead, we note that $A \subseteq A$, so $A \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $A \in \mathcal{P}(B)$. Thus $A \subseteq B$. That is, if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

We have proved both implications, so we're done.

Problem 4 If possible, find sets A and B and a function $f : A \to B$ such that:

a. $\aleph_0 \leq \overline{\overline{A}} = \overline{\overline{B}}$ and f is onto, but not 1-1

$f : \mathbf{R} \longrightarrow [0, \infty)$ by $f(x) = x^2$ is onto but not 1-1; both the domain and codomain have cardinality $\mathbf{c}$.

b. $\overline{\overline{A}} = \overline{\overline{B}} < \aleph_0$ and f is 1-1, but not onto

Not possible. Suppose A and B have the same *finite* cardinality, and $f : A \xrightarrow{\text{1-1}} B$. Since f is onto its range, $f : A \xrightarrow[\text{onto}]{\text{1-1}} f(A) \subseteq B$. Thus $\overline{\overline{f(A)}} = \overline{\overline{A}} = \overline{\overline{B}}$. If f is not onto, then $f(A)$ is a proper subset of B, and cardinally equivalent to B, contradicting the fact that a finite set is not equivalent to one of its proper subsets. Thus f must be onto B.

c. A is uncountable and B is finite

Any constant function will do: say, $f : \mathbf{R} \longrightarrow \{1\}$ by $f(x) = 1$ for all $x \in \mathbf{R}$.

d. A is denumerable, B is uncountable, and f is onto.

Not possible. If $f : A \xrightarrow{\text{onto}} B$, then $\overline{\overline{A}} \geq \overline{\overline{B}}$.

Problem 5 For $n \in \mathbf{N}$, let $A_n = [1 - \frac{1}{n}, 2 - \frac{1}{n}] \cup \{n\}$. Find:

a.

$$\bigcap_{n=1}^{\infty} A_n$$

The intersection is $\{1\}$.

Proof (optional): First, note that for any $n \in \mathbf{N}$, $0 \leq n - 1$, so (adding n to both sides) $n \leq 2n - 1$. Also, $n - 1 \leq n$. Thus, we have $n - 1 \leq n \leq 2n - 1$. Dividing through by n, we find that $1 - \frac{1}{n} \leq 1 \leq 2 - \frac{1}{n}$, so $1 \in A_n$. Since 1 belongs to every A_n, it belongs to their intersection.

Now we will see that no other number lies in the intersection. Consider a real number $x \neq 1$. If $x < 1$, then let $\varepsilon = 1 - x > 0$. It follows that $x = 1 - \varepsilon$. Let $n = \lceil \frac{1}{\varepsilon} \rceil + 1$, so that n is a natural number and $n > \frac{1}{\varepsilon}$. Now $\frac{1}{n} < \varepsilon$, so $-\frac{1}{n} > -\varepsilon$; thus, $1 - \frac{1}{n} > 1 - \varepsilon = x$. That is, x lies to the left of the left endpoint of $[1 - \frac{1}{n}, 2 - \frac{1}{n}]$, so $x \notin A_n$. Thus x does not belong to the intersection. On the other hand, if $x > 1$, then $x \notin A_1$, because $A_1 = [0, 1]$. Either way, if $x \neq 1$, then for some natural number n, $x \notin A_n$.

b.

$$\bigcup_{n=1}^{\infty} A_n$$

The union is $[0, 2) \cup \mathbf{N}$.

Proof (optional): To show this, we must demonstrate that every number in $[0, 2) \cup \mathbf{N}$ lies in some A_n. We know that any natural number n belongs to A_n (because of that "$\cup\{n\}$" at the end of the definition of A_n), so we must only prove it for $x \in [0, 2)$. As remarked earlier, $A_1 = [0, 1]$, so in fact we are already done except in the case $1 < x < 2$. In this case, let $\varepsilon = 2 - x > 0$. Choose some $n > \frac{1}{\varepsilon}$, such as the value $n = \lceil \frac{1}{\varepsilon} \rceil + 1$ we used earlier; now $\frac{1}{n} < \varepsilon$. We have assumed that $1 < x = 2 - \varepsilon$, so it follows that $1 < x + \frac{1}{n} = 2 - \varepsilon + \frac{1}{n} = 2 + (\frac{1}{n} - \varepsilon) < 2$. That is, $1 < x + \frac{1}{n} < 2$, so $1 - \frac{1}{n} < x < 2 - \frac{1}{n}$. That is, $x \in A_n$. We have just shown that $[0, 2) \cup \mathbf{N} \subseteq \cup_{n=1}^{\infty} A_n$.

For the reverse containment, note that $A_n \subseteq [0, 2) \cup \mathbf{N}$ for all n; thus, their union is also a subset of $[0, 2) \cup \mathbf{N}$. Since we have proved both containments, it follows that $\cup_{n=1}^{\infty} A_n = [0, 2) \cup \mathbf{N}$.

c. The symmetric difference $A_1 \Delta A_3$

Since $A_1 = [0,1]$ and $A_3 = [\frac{2}{3}, \frac{5}{3}] \cup \{3\}$, we may compute their union and intersection:

- $A_1 \cup A_3 = [0, \frac{5}{3}] \cup \{3\}$
- $A_1 \cap A_3 = [\frac{2}{3}, 1]$
- $$\begin{aligned} A_1 \Delta A_3 &= (A_1 \cup A_3) - (A_1 \cap A_3) \\ &= ([0, \frac{5}{3}] \cup \{3\}) - ([\frac{2}{3}, 1]) \\ &= [0, \frac{2}{3}) \cup (1, \frac{5}{3}] \cup \{3\} \end{aligned}$$

Problem 6 Let $\mathcal{A}$ be a partition of A. Define the relation Q on A by: xQy iff for some $S \in \mathcal{A}$, $x \in S$ and $y \in S$. Prove that Q is reflexive. Let $x \in A$; we must show that xQx. Since $\mathcal{A}$ is a partition of A, we know that $\cup_{S \in \mathcal{A}} S = A$. That is, $x \in \cup_{S \in \mathcal{A}} S$, so for some $S \in \mathcal{A}$, $x \in S$. Thus $x \in S$ and $x \in S$, so xQx.

Problem 7 Prove or disprove:

a. For all sets A, B, and C: If $A - B \subseteq C$, then $C - B \subseteq A$.

False. If $A = \emptyset$ and $C - B \neq \emptyset$, then $A - B = \emptyset \subseteq C$, while $C - B$ is too big to be a subset of A. (Every subset of A is empty.)

b. For all sets A, B, and C: If $A - B \subseteq C$, then $A \subseteq B \cup C$.

True. Suppose $A - B \subseteq C$, and let $x \in A$. If $x \in B$, then $x \in B \cup C$. If $x \notin B$, then $x \in A$ and $x \notin B$; that is, $x \in A - B$. Since $A - B \subseteq C$, we conclude that $x \in C$, so $x \in B \cup C$. In all cases, if $x \in A$, then $x \in B \cup C$. So, $A \subseteq B \cup C$.

Alternative proof: For sets U, V, X, and Y: if $U \subseteq V$ and $X \subseteq Y$, then $U \cup X \subseteq V \cup Y$. Thus, $[(A - B) \cup B] \subseteq C \cup B$. But $[(A - B) \cup B] = (A \cap \tilde{B}) \cup B = (A \cup B) \cap (\tilde{B} \cup B) = A \cup B$, and $A \subseteq A \cup B$, so $A \subseteq B \cup C$.

Problem 8 Let S be a fixed subset of $\mathbf{R}$. Define the relation $\star$ on $\mathcal{P}(\mathbf{R})$ by $A \star B$ iff $A - S = B - S$.

a. Prove that $\star$ is an equivalence relation.

Let A, B, and C be arbitrary subsets of $\mathbf{R}$.

$\star$ is reflexive: $A - S = A - S$, so $A \star A$.

$\star$ is symmetric: If $A \star B$, then $A - S = B - S$; thus $B - S = A - S$ and $B \star A$.

$\star$ is transitive: IF $A \star B$ and $B \star C$, then $A - S = B - S$ and $B - S = C - S$. Thus $A - S = C - S$, so $A \star C$.

b. ♣ If $S = \{0, 1\}$ and $A = [0, 1]$, describe $A/\star$.

$A \star B$ basically means that A and B appear to be the same set if we ignore elements of S. In this case, $A/\star$ is the set of all intervals from 0 to 1, whether they be open, closed, or half-open. We already have $A = [0, 1]$; if we also define $B = [0, 1)$, $C = (0, 1]$, and $D = (0, 1)$, then $A - S = B - S = C - S = D - S = (0, 1)$.

Since $\overline{\overline{S}} = 2$, $\overline{\overline{A/\star}} = 2^2$. If we had chosen S to have 3 elements, then $A/\star$ would have 8. If you're curious, take $S = (0, 1)$ or $S = \mathbf{N}$ and ponder the cardinality of $A/\star$.

Problem 9 Let $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3 - x$. Find:

a. $f([-1, 1])$

This is a calculus question. Since $f(x)$ is continuous, I'm essentially asking you to find the extreme values of f on the interval [-1,1]. Note that if you just try $x = \pm 1$, you find $f(x) = 0$. Zero is not the answer. We must locate all relative extrema of the function. To do this, we take the derivative: $f'(x) = 3x^2 - 1$. We set this to zero and solve for x:

$$\begin{aligned} 3x^2 - 1 &= 0 \\ 3x^2 &= 1 \\ x^2 &= \frac{1}{3} \\ x &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

As you can confirm, if $x = \pm 1/\sqrt{3}$, then $f(x) = \mp 2/3\sqrt{3}$. That is, the extreme values are $\pm 2/3\sqrt{3}$, so $f([-1, 1]) = [-2/3\sqrt{3}, 2/3\sqrt{3}]$.

b. $f^{-1}(\{0\})$

The preimages of 0 are the roots of the polynomial; $f^{-1}(\{0\}) = \{-1, 0, 1\}$.

c. A set S such that $f|_S$ is 1-1 and onto $\mathbf{R}$

The two bumps cause us some trouble with the 1-1 business, so let's get rid of them. Let $S = (-\infty, -1) \cup \{0\} \cup (1, \infty)$. Note that we can include only one of the roots; its doesn't really matter which, I just picked 0 because it was in the middle.

Problem 10 Use the Principle of Mathematical Induction to prove the following formula for the sum of a geometric series:

$$\sum_{k=0}^{n} a \cdot r^k = a\left(\frac{1 - r^{n+1}}{1 - r}\right)$$

Er... actually, the formula requires $r \neq 1$. We'll prove it in that case.
Note that if $n = 1$, the claim says that $a \cdot r^0 + a \cdot r^1 = a\left(\frac{1-r^2}{1-r}\right)$. This is true because $\frac{1-r^2}{1-r} = 1 + r$. That is, $a\left(\frac{1-r^2}{1-r}\right) = a(1 + r) = a + ar = a \cdot r^0 + a \cdot r^1$.

Now suppose the claim holds for some number n; that is, suppose

$$\sum_{k=0}^{n} a \cdot r^k = a\left(\frac{1-r^{n+1}}{1-r}\right).$$

It follows that

$$\begin{aligned}
\sum_{k=0}^{n+1} a \cdot r^k &= \left(\sum_{k=0}^{n} a \cdot r^k\right) + a \cdot r^{n+1} \\
&= a\left(\frac{1-r^{n+1}}{1-r}\right) + a \cdot r^{n+1} \\
&= a\left(\frac{1-r^{n+1}}{1-r} + \frac{r^{n+1}(1-r)}{1-r}\right) \\
&= a\left(\frac{1-r^{n+1}}{1-r} + \frac{r^{n+1}-r^{n+2}}{1-r}\right) \\
&= a\left(\frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r}\right) \\
\sum_{k=0}^{n+1} a \cdot r^k &= a\left(\frac{1-r^{n+2}}{1-r}\right)
\end{aligned}$$

That is, whenever the claim holds for n, it holds for $n+1$ as well. By PMI, it is true for all $n \in \mathbb{N}$.

Problem 11 ♣ Use the WOP to prove that every natural number is interesting. Let T be the set of uninteresting natural numbers. If T is nonempty, then by WOP, T contains a least element m. Then m is *the smallest uninteresting natural number*—which is a very interesting thing to be! Thus, $m \notin T$ afterall, which is a contradiction. Therefore $T = \emptyset$, and every natural number is interesting.

Problem 12 Let $a_1 = 1$, $a_2 = 4$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n > 2$. Use PCI or WOP to prove that, for every natural number n, $a_n = n \cdot 2^{n-1}$. PCI: Suppose n be a natural number such that, for all $m < n$, $a_m = m \cdot 2^{m-1}$. We will show that $a_n = n \cdot 2^{n-1}$. If $n = 1$, then $a_n = a_1 = 1 = 1 \cdot 2^0 = n \cdot 2^{n-1}$, so the claim is true for n.
If $n = 2$, then $a_n = a_2 = 4 = 2 \cdot 2^1 = n \cdot 2^{n-1}$, so again the claim is true.
If $n > 2$, then $n-1$ and $n-2$ are natural numebrs less than n; thus, by our inductive hypothesis, $a_{n-1} = (n-1) \cdot 2^{n-1-1}$ and $a_{n-2} = (n-2) \cdot 2^{n-2-1}$. That is, $a_{n-1} = (n-1) \cdot 2^{n-2}$ and $a_{n-2} = (n-2) \cdot 2^{n-3}$. Now

$$\begin{aligned}
a_n &= 4a_{n-1} - 4a_{n-2} \\
&= 4(n-1) \cdot 2^{n-2} - 4(n-2) \cdot 2^{n-3} \\
&= 2(n-1) \cdot 2^{n-1} - (n-2) \cdot 2^{n-1} \\
&= [(2n-2) - (n-2)] \cdot 2^{n-1} \\
a_n &= n \cdot 2^{n-1}
\end{aligned}$$

So, the claim holds for n. By PCI, the claim is true for all natural numbers.

Problem 13 Prove that the set $[1,2)\cup(3,4)$ has cardinal number $\mathbf{c}$. Note that $(3,4)\subseteq[1,2)\cup(3,4)\subseteq(0,5)$. By a theorem in the book, $\overline{\overline{(3,4)}}=\overline{\overline{(0,5)}}=\mathbf{c}$. Thus, $\mathbf{c}\leq\overline{\overline{[1,2)\cup(3,4)}}\leq\mathbf{c}$. By Cantor-Schröder-Bernstein, $\overline{\overline{[1,2)\cup(3,4)}}=\mathbf{c}$.

Problem 14 Repeat problem 13, without using Cantor-Schröder-Bernstein. (That is, construct a bijection.) Let $f_1=\mathrm{Id}_{[1,2)}$. That is, let $f_1:[1,2)\longrightarrow[1,2)$ by $f(x)=x$. (Then f_1 is a bijection.)
Let $f_2:(3,4)\longrightarrow(0,1)$ by $f(x)=x-3$. Then f_2 is also a bijection. It is 1-1 because: if $f(x)=f(y)$, then $x-3=y-3$, so $x=y$. It is onto because: if $z\in(0,1)$, then $z+3\in(3,4)$ and $f_2(z+3)=z+3-3=z$.
Since $f_1:[1,2)\xrightarrow[\text{onto}]{\text{1-1}}[1,2)$ and $f_2:(3,4)\xrightarrow[\text{onto}]{\text{1-1}}(0,1)$ and $[1,2)\cap(3,4)=\emptyset$ and $[1,2)\cap(0,1)=\emptyset$, it follows that $f_1\cup f_2:[1,2)\cup(3,4)\xrightarrow[\text{onto}]{\text{1-1}}[1,2)\cup(0,1)$. That is, $f_1\cup f_2:[1,2)\cup(3,4)\xrightarrow[\text{onto}]{\text{1-1}}(0,2)$. And we know (by a theorem) that $(0,2)$ has cardinal number $\mathbf{c}$; thus, $[1,2)\cup(3,4)$ does, too.

Note: Problems marked with the club (♣) are somewhat wacky. If they don't make sense to you, don't despair.