Practice Midterm 2 Solutions

Problem 1 *(10 points)*
True or False? Circle one.

T/F For any functions F and G, $\text{Dom}(G \circ F) \subseteq \text{Dom}(F)$.

True. $G \circ F = \{(x, z) : \text{for some } y,\ (x, y) \in F \text{ and } (y, z) \in G\}$. Therefore, $\text{Dom}(G \circ F)$ is the set of all first coordinates appearing in that set, namely:

$$\begin{aligned}
\text{Dom}(G \circ F) &= \{x : \text{for some } z, (x, z) \in G \circ F\} \\
&= \{x : \text{for some } y \text{ and } z,\ (x, y) \in F \text{ and } (y, z) \in G\} \\
&= \{x : \text{for some } y,\ (x, y) \in F \text{ and for some } z,\ (y, z) \in G\} \\
&= \{x : \text{for some } y,\ (x, y) \in F \text{ and (some other condition)}\} \\
&\subseteq \{x : \text{for some } y,\ (x, y) \in F\} \ = \ \text{Dom}(F)
\end{aligned}$$

T/F For any functions F and G, $\text{Dom}(G \circ F) \subseteq \text{Dom}(G)$.

False. We've already seen (above) that $\text{Dom}(G \circ F) \subseteq \text{Dom}(F)$. If F and G are functions with disjoint domains, then the domain of $G \circ F$ must also be disjoint from $\text{Dom}(G)$.

T/F Every relation that is symmetric and transitive is also reflexive.

False. The empty relation on A, where A is whatever nonempty set you like, is symmetric and transitive, but not reflexive.

T/F The relation $\{(1, 2), (2, 3), (3, 1)\}$ is transitive.

False. Since the relation contains $(1, 2)$ and $(2, 3)$, it must contain $(1, 3)$ in order to be transitive.

Problem 2 *(10 points)*
There is a natural correspondence between equivalence relations and partitions. If A is a set and R is an equivalence relation on A, then A/R, the set of R's equivalence classes, is a partition of A. Prove or disprove: If R and S are two equivalence relations on A, then $A/R \cap A/S$ is a partition of A.

The claim is false. Let A be the set $\{1, 2, 3, 4\}$, and define relations R and S on A as follows: xRy iff $x \equiv_2 y$, and xSy iff $(x - 2.5)(y - 2.5) > 0$. That is:
$R = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (2, 4), (4, 2), (4, 4)\}$ and
$S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$.
(You could also describe R and S by way of a digraph, if you like.) The point is,
$A/R = \{\{1, 3\}, \{2, 4\}\}$ while $A/S = \{\{1, 2\}, \{3, 4\}\}$. I'd like to point out that $\{1, 3\} \in A/R$ but $\{1, 3\} \notin A/S$. You can check each element; we find that $A/R \cap A/S = \varnothing$. And $\varnothing$ is not a partition of A.

Problem 3 *(10 points)*
Prove that the relation "$\equiv_7$" is an equivalence relation.

To do this, we must know that $x \equiv_7 y$ iff $7|x - y$, and that the domain is assumed to be $\mathbf{Z}$. Now we prove three things:

1. $\equiv_7$ is reflexive: For any $x \in \mathbf{Z}$, $x - x = 0 = 0 \cdot 7$, so $7|x - x$. Thus, $x \equiv_7 x$.

2. $\equiv_7$ is symmetric. Suppose $x \equiv_7 y$. Then $7|x - y$, so $7| - (x - y)$, which is to say that $7|y - x$. Thus $y \equiv_7 x$.

3. $\equiv_7$ is transitive. Suppose $x \equiv_7 y$ and $y \equiv_7 z$. Then $7|x - y$ and $7|y - z$, so $7|(x - y) + (y - z)$. That is, $7|x - z$, so $x \equiv_7 z$.

Since $\equiv_7$ is reflexive, symmetric, and transitive, it is an equivalance relation.

Problem 4 *(10 points)*
Consider the relation R on $\mathbf{N}$, defined by: xRy iff x and y have a common factor greater than 1. Is this relation reflexive? Symmetric? Transitive?

This relation is *almost* reflexive, but not quite. After all, it is not true that $1R1$.
This relation is symmetric: if x and y have a common factor $k > 1$, then y and x have the common factor k.
The relation is not transitive: $2R6$ and $6R3$, but not $2R3$.

Problem 5 *(10 points)*
Prove that if $f : A \overset{1-1}{\to} B$ and $g : B \overset{1-1}{\to} C$, then $g \circ f$ is 1-1. Does this imply that $g \circ f : A \overset{1-1}{\to} C$?

Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. This implies that $g(f(x_1)) = g(f(x_2))$. Since g is 1-1, it follows that $f(x_1) = f(x_2)$; now, since f is 1-1, we conclude that $x_1 = x_2$. Thus $g \circ f$ is 1-1.
Furthermore, for any $x \in A$, $f(x) \in B = \mathrm{Dom}(g)$, so $g(f(x)) \in C$. This means $\mathrm{Dom}(g \circ f) = A$, and so $g \circ f : A \overset{1-1}{\to} C$.
However, there is a related question that turns out differently. If $f : A \overset{1-1}{\to} B$ and $g : C \overset{1-1}{\to} D$, it will still be the case that $g \circ f$ is 1-1, and it will also be true that $g \circ f \subseteq A \times D$. But, it does not necessarily follow that $g \circ f : A \overset{1-1}{\to} D$, because $\mathrm{Dom}(g \circ f)$ may be a proper subset of A. For instance, if $f(x) = x - 1$ for all $x \in \mathbf{R}$ and $g(x) = \log x$ for all $x > 0$, then $(g \circ f)(x) = \log(x - 1)$ for all $x > 1$. The domain of f is $\mathbf{R}$, but the domain of $g \circ f$ is only half as big: $(1, \infty)$. We could say $g \circ f : (1, \infty) \overset{1-1}{\to} \mathbf{R}$, but not $g \circ f : \mathbf{R} \overset{1-1}{\to} \mathbf{R}$.

Problem 6 *(10 points)*
Let $f(x) = \frac{2x+1}{x-1}$. Find the domain of f, assuming that it is the largest possible subset of $\mathbf{R}$. Find the range of f. Find an expression for $f^{-1}(x)$. What are the domain and range of f^{-1}?

The domain of f is $\mathbf{R}-\{1\}$. (A rational function is defined at any real number for which the denominator is nonzero.) Now let's find the range. Note that a real number y is in the range of f iff there is some x such that $y = f(x)$; so, we'll try to decide when such an x exists. We'll solve the equation for x.

$$\begin{aligned} y &= \frac{2x+1}{x-1} \\ (x-1)y &= 2x+1 \\ xy - y &= 2x+1 \\ xy - 2x &= y+1 \\ x(y-2) &= y+1 \\ x &= \frac{y+1}{y-2} \end{aligned}$$

We see that for every y except $y = 2$, this equation tells us a preimage (really, *the* preimage) of y. That is, 2 is the only real number without a preimage, so $\mathrm{Rng}(f) = \mathbf{R} - \{2\}$.

To find an expression for $f^{-1}(x)$, we can simply exchange the roles of x and y in the final equation above, getting $y = \frac{x+1}{x-2}$. That is, $f^{-1}(x) = \frac{x+1}{x-2}$. Now $\mathrm{Dom}(f^{-1}) = \mathrm{Rng}(f) = \mathbf{R} - \{2\}$ and $\mathrm{Rng}(f^{-1}) = \mathrm{Dom}(f) = \mathbf{R} - \{1\}$.

Problem 7 *(10 points)*

Find an example of a function $f : A \to B$ that is 1-1, and a function $g : B \to C$ that is NOT 1-1, such that $g \circ f$ is 1-1, or prove that this is impossible.

Two such functions are $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = e^x$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = x^2$. Then f is 1-1 and g is not, but the composition is 1-1: $(g \circ f)(x) = g(f(x)) = (e^x)^2 = e^{2x}$. The reason this works is that, while g is not 1-1, it becomes 1-1 when we restrict its domain to $[0, \infty)$. If f is an injection into this smaller set, then the composition is injective. You could think of it this way: The reason g is not 1-1 is that every positive number, such as 2, has an "evil twin" (-2, in this case), meaning $g(2) = 2^2 = 4 = (-2)^2 = g(-2)$. So, the distinct inputs 2 and -2 have the same output, making g noninjective. But the function f has range $(0, \infty)$, so it never wakes up the evil twins.