Angle Sum Identities

This is one way to understand the angle sum identities

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and

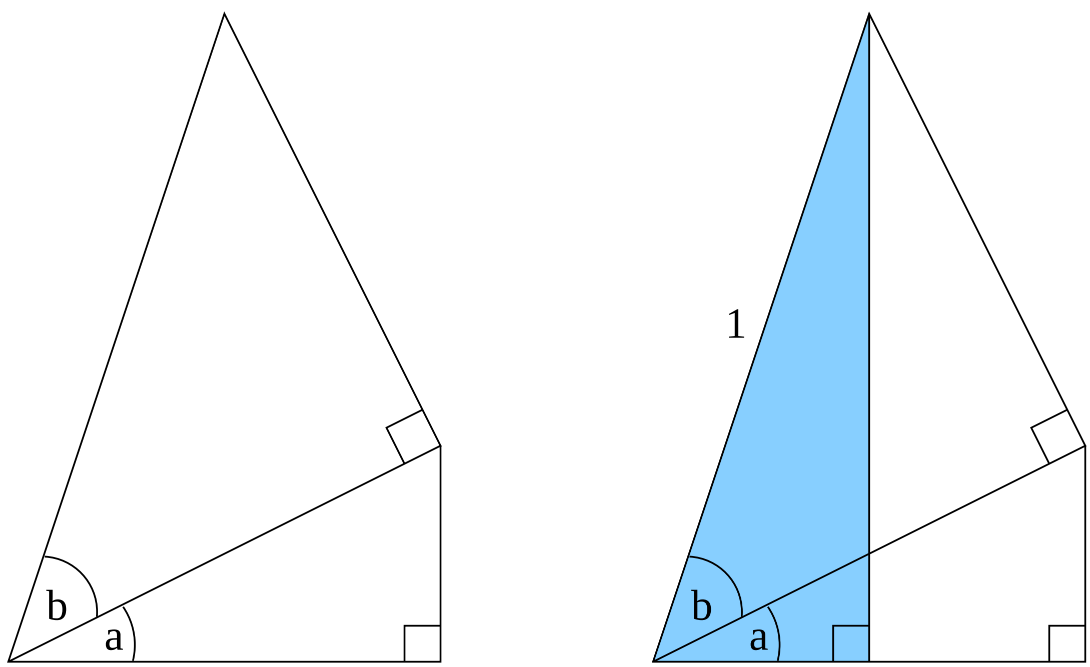
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

It makes frequent use of these side length rules:

$$\text{Since } \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \text{ we have } \boxed{\text{Adjacent} = \text{Hypotenuse}\cdot\cos\theta}$$

$$\text{Since } \sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \text{ we have } \boxed{\text{Opposite} = \text{Hypotenuse}\cdot\sin\theta}$$

We start with two triangles, stacked like this:



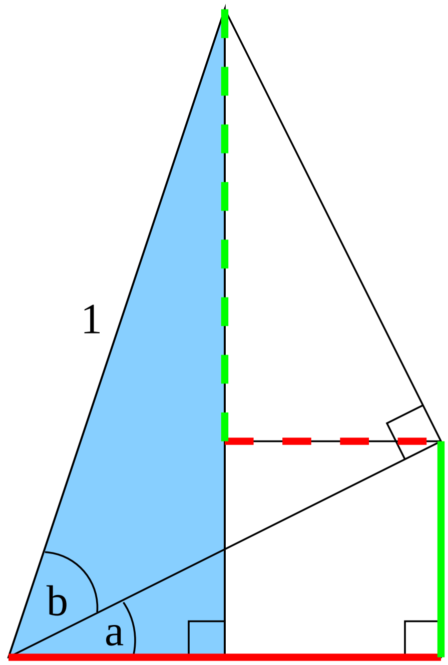
Notice that the total base angle is $a+b$. We could choose the scale to be whatever we want; it will be most convenient if we make the "top" hypotenuse 1. If we do that, then:

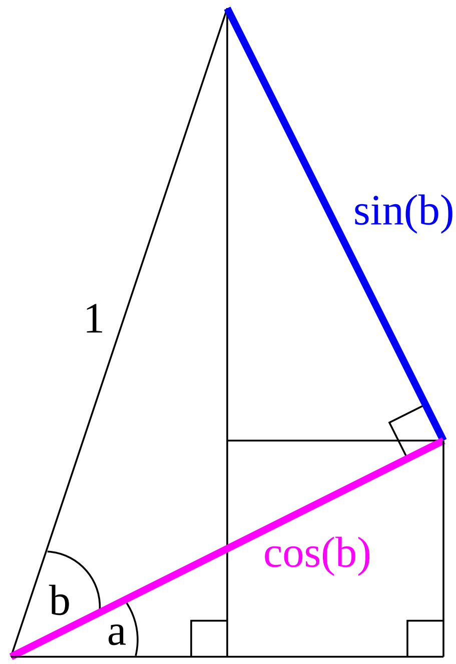
- the Opposite Side Rule says the blue triangle above has height $1\cdot\sin(a+b) = \sin(a+b)$, and
- the Adjacent Side Rule says the blue triangle above has width $1\cdot\cos(a+b) = \cos(a+b)$.

Now our goal is to write that height and width in a different way, using the triangles with base angles a and b.

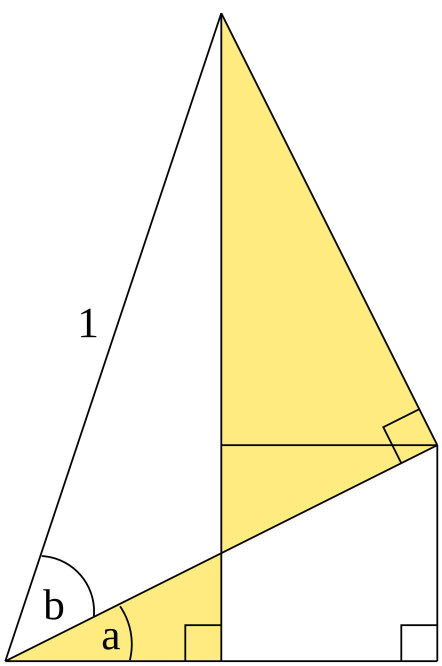
It helps to add an extra horizontal line to the picture; see the diagram below. If we add the green lengths, we get the height of the blue triangle; if we subtract the length of the dashed red line from the length of the solid red line, we get the width of the blue triangle. But let's deal with that later.

Now let's start figuring out all the relevant side lengths in the diagram, using the Opposite Side Rule and the Adjacent Side Rule. First, we work on the triangle with base angle b, which has hypotenuse 1. By the Rules, we know its opposite side has length $\sin(b)$, and its adjacent side has length $\cos(b)$:

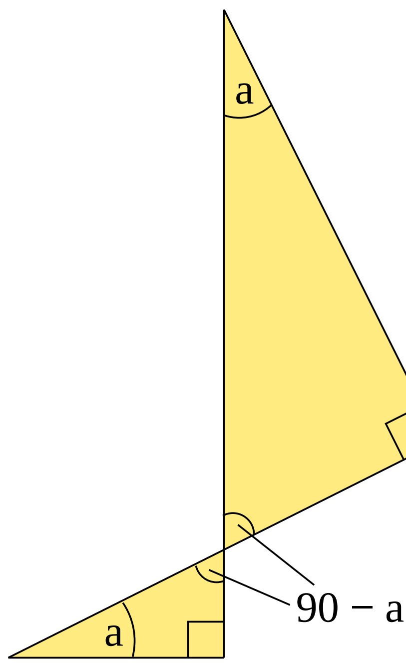


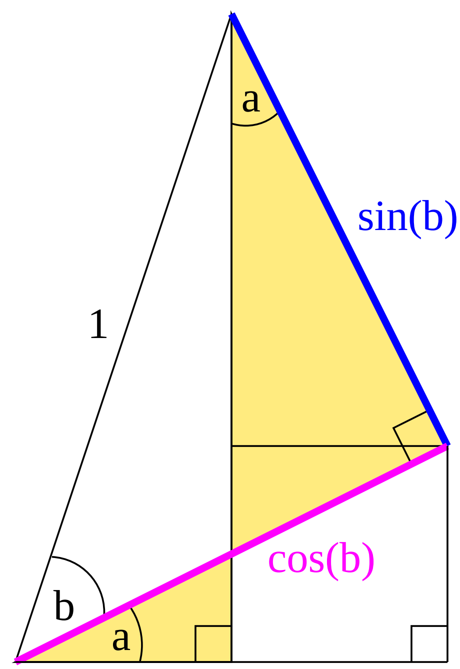


These two sides form the hypotenuses of two smaller right triangles. Before we can make further progress with those triangles, we need to know the angle between the blue edge and the vertical. So, we consider the following bow-tie shape:

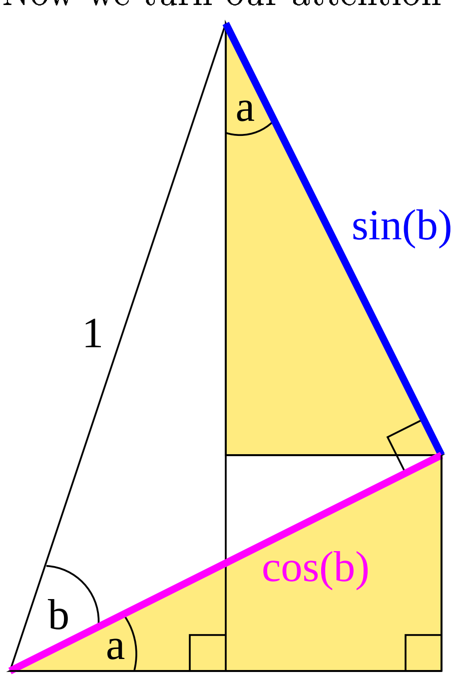


Here we pull the bow-tie out of the triangles altogether so we can get a good look at it. When a right triangle has base angle a, the other angle is the *complement* of a. In degrees, that's $90 - a$; in radians, that's $\frac{\pi}{2} - a$. Whatever, we're not really going to use that angle very much so let's not dwell on it. Where the two gold triangles meet (at the middle of the bow tie), their edges form an X. Angles that are opposite each other in an X are called "vertical angles". More importantly, they are congruent. If one of them measures $90 - a$, so does the other. Now, the mystery angle in the top corner is the complement of the complement of a; that is, it measures $90 - (90 - a) = a$. It's the same as base angle a. Let's label that in the bow-tie and in our Big Diagram, and get back to business.

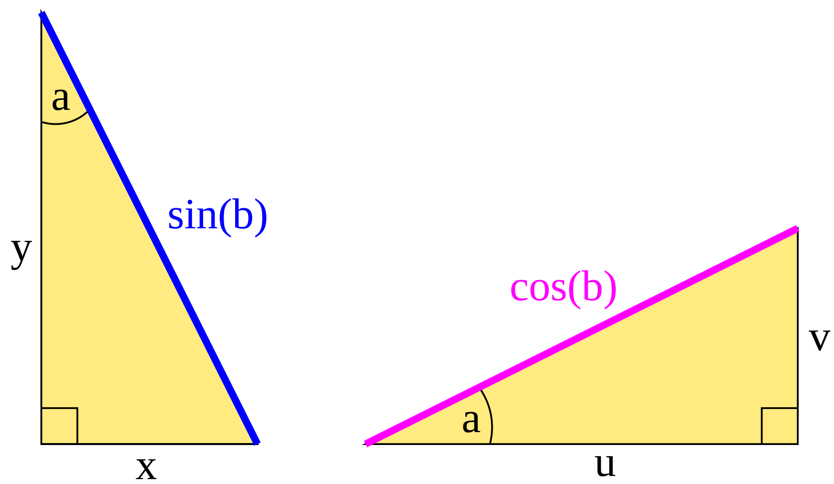




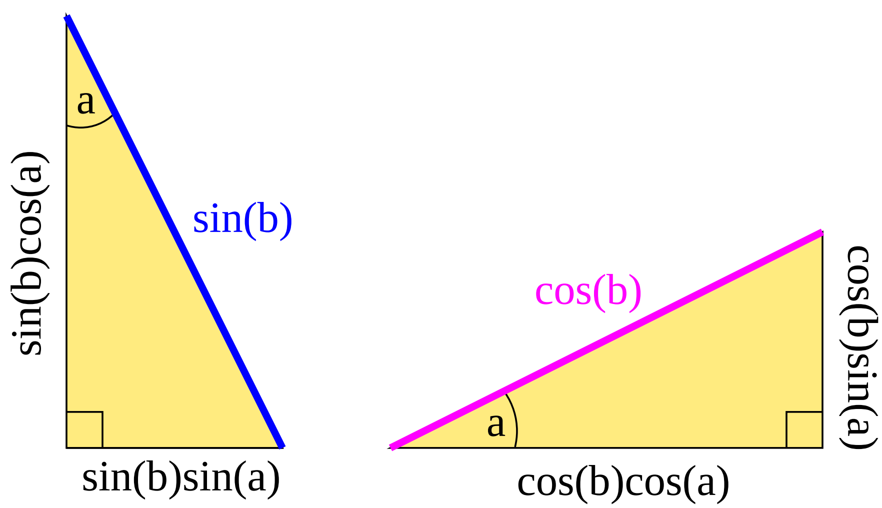
Now we turn our attention to the triangles with blue and magenta hypotenuses:



Let's pull those two triangles out of the Big Diagram so we can see them more clearly. Here they are:

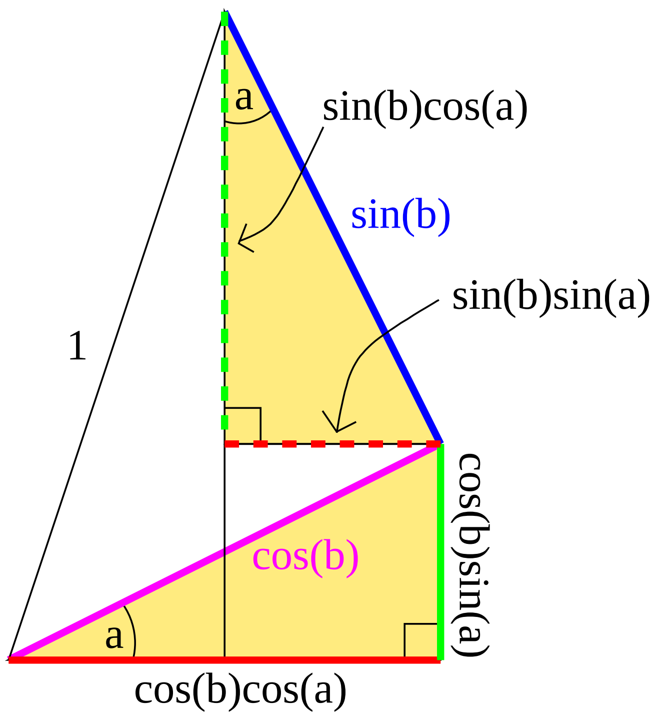


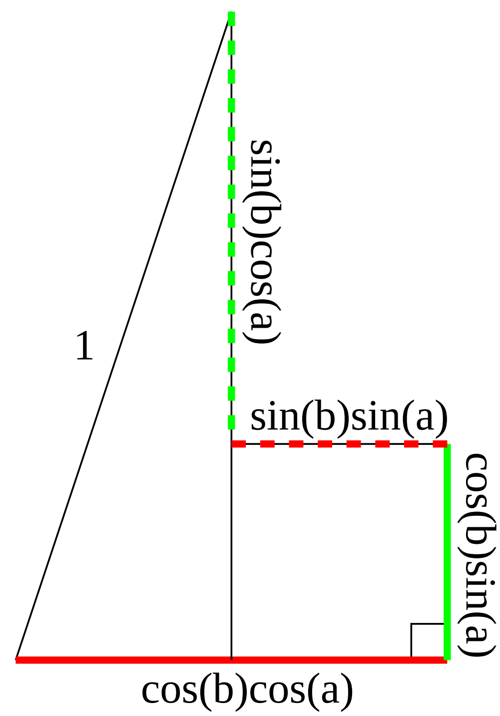
To find the height of the first one, we notice that the height of that triangle is adjacent to the angle a. So, we use the Adjacent Side Rule, and find that the height y is the hypotenuse $\sin(b)$ times $\cos(a)$, or $\boxed{y = \sin(b)\cos(a)}$. Since x is the side opposite that angle a, the Opposite Side Rule tells us that $\boxed{x = \sin(b)\sin(a)}$. We compute the lengths of the second triangle's legs in the same way; we find $\boxed{u = \cos(b)\cos(a)}$ and $\boxed{v = \cos(b)\sin(a)}$. Let's replace x, y, u, and v with these values in the picture:



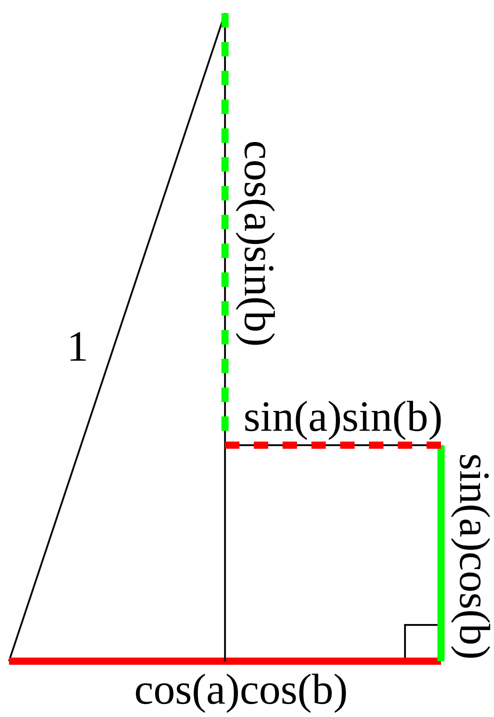
On the next page, we put these triangles back into the Big Diagram.

In the first figure below, we've put the two triangles back into the Big Diagram. All we need from them is the lengths of their legs. Once we have those lengths properly labeled, we can forget about the triangles.

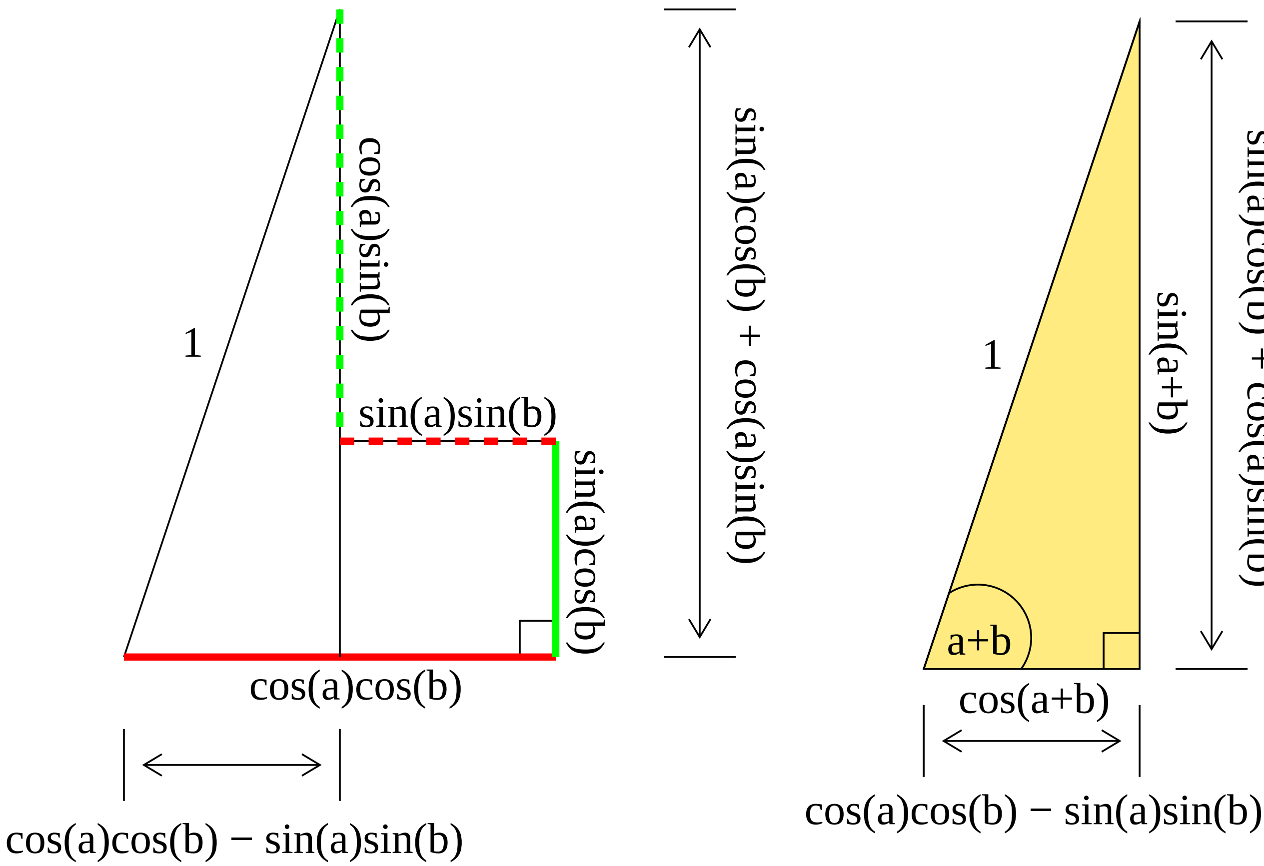




Notice how we've found the lengths of all the red and green edges. Now let's clean up those labels a little. I'd like to get the angles to appear in alphabetical order, for consistency; so, I'm rewriting $\cos(b)\sin(a)$ as $\sin(a)\cos(b)$, etc.



Now I'd like to emphasize that the total height of the diagram is the sum of the green lengths. Also, I've marked a certain width in the picture that is the difference of the red lengths. Recall from Page 2 that the height of the gold triangle is $\sin(a+b)$, and its width is $\cos(a+b)$. We've found two different ways of expressing the same lengths:



From this, we get the angle sum identities

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

With these, we can derive the double angle identities (set $a = \theta$, $b = \theta$); from there, we can derive the half angle identities. The angle sum identities can also be used to build the sum-to-product and product-to-sum formulas, which are not actually part of this class.

I would like to use them now to build the angle sum identity for tangent. We start with the idea that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. So:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} \\
&= \frac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\cos(a)\cos(b)-\sin(a)\sin(b)} \\
& \quad \text{Divide top and bottom by } \cos(a)\cos(b). \\
&= \frac{\frac{\sin(a)\cos(b)}{\cos(a)\cos(b)}+\frac{\cos(a)\sin(b)}{\cos(a)\cos(b)}}{\frac{\cos(a)\cos(b)}{\cos(a)\cos(b)}-\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}} \\
&= \frac{\frac{\sin(a)}{\cos(a)}+\frac{\sin(b)}{\cos(b)}}{1-\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}} \\
\tan(a+b) &= \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}
\end{aligned}$$