Angle Sum Identities

You've seen the previous link, about the Angle Sum formulas. Maybe you followed it, or maybe it was overwhelming. Now I'd like to show you the same angle sum formulas, proved using the following mysterious fact about complex numbers:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where $i = \sqrt{-1}$ is the famous imaginary number. I'll explain later why this is true...but for the time being, I'd like to you just trust me on this, strange as it may look.

Let's replace θ in the above formula with $\alpha + \beta$, so we get:

$$e^{i(\alpha+\beta)} = \boxed{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}.$$

By computing the left-hand side in a different way, we find:

$$\begin{aligned}
e^{i(\alpha+\beta)} &= e^{i\alpha+i\beta} \\
&= e^{i\alpha}e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\
&\qquad \text{(using the mysterious fact twice)} \\
&= \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + i^2\sin\alpha\sin\beta \quad \text{(FOIL)} \\
&= \cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta - \sin\alpha\sin\beta \\
&\qquad \text{(because } i^2 = -1\text{)} \\
&= \boxed{(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)} \\
&\qquad \text{(grouping real and imaginary terms together)}
\end{aligned}$$

Now we have two different expressions for $e^{i(\alpha+\beta)}$, so they must actually be different ways of saying the same thing. That is:

$$\boxed{\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}$$

If you think that's ugly, look again. In order for two complex numbers to be equal, they must have the same real part and the same imaginary part; that is,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Both angle sum formulas appeared as the result of a single computation! If you made an earnest attempt at understanding the previous description, you've got to appreciate how much simpler this one was. And they call these numbers complex!