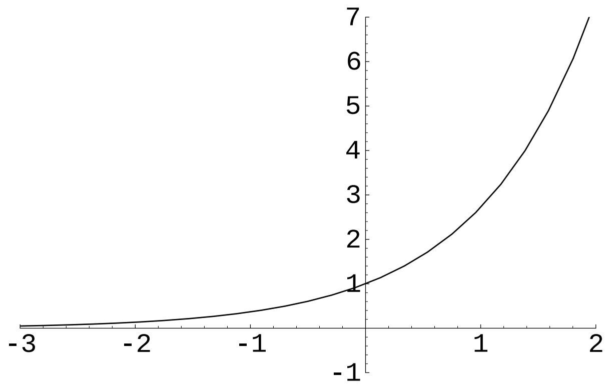
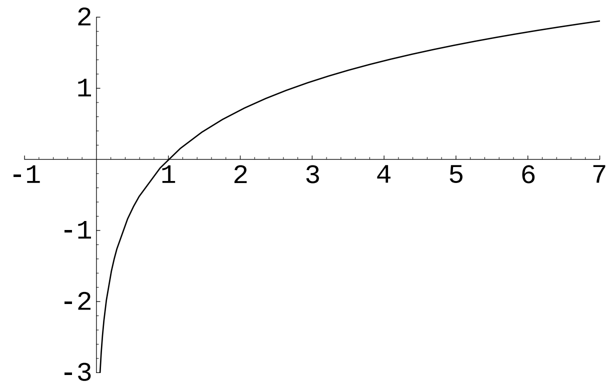
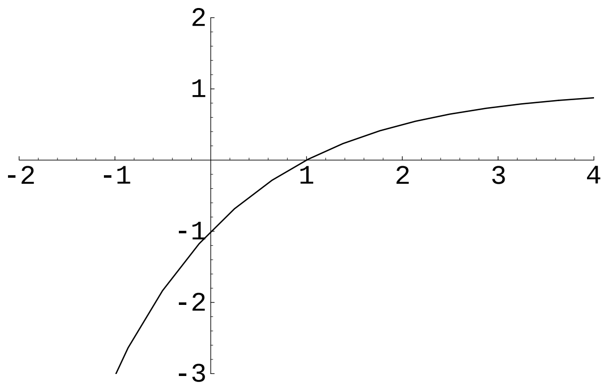
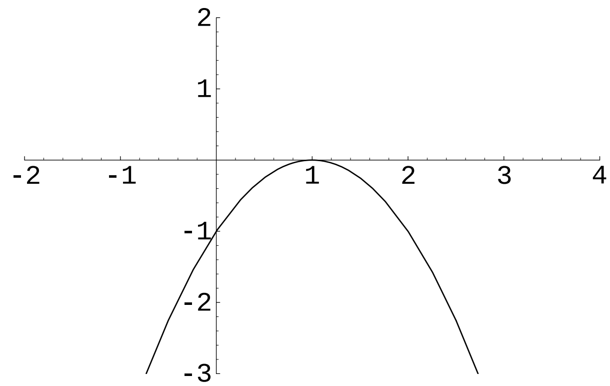
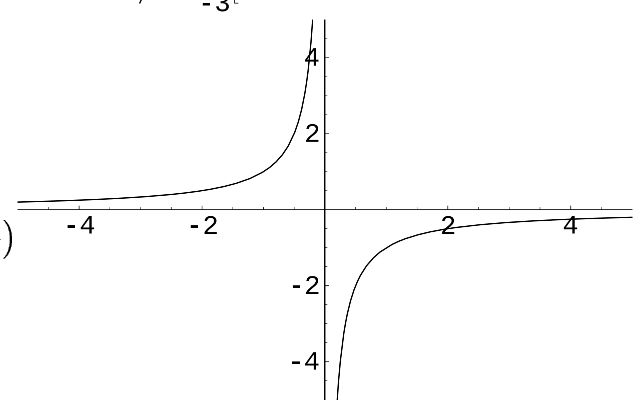
Problem 1 *(10 points)*
Match these graphs to their functions.

A)


B)


C)


D)


E)


[C] $y = 1 - 2(2^{-x})$
[A] $y = e^x$
[E] $y = -1/x$
[B] $y = \ln x$
[D] $y = -(x-1)^2$

Problem 2 *(10 points)*
What is the average rate of change of $y = \sin x$ on the interval $[\pi/2, \pi]$?

$$\text{Average rate of change} = \frac{\sin\pi - \sin(\pi/2)}{\pi - (\pi/2)} = \frac{0-1}{\pi/2} = -\frac{2}{\pi}.$$

What is the average rate of change of $y = \cos x$ on the interval $[0, \pi/2]$?

$$\text{Average rate of change} = \frac{\cos(\pi/2) - \cos 0}{(\pi/2) - 0} = \frac{0-1}{\pi/2} = -\frac{2}{\pi}.$$

Problem 3 *(10 points)*
Find the minimum value of $y = x^4 + 2x^2 + 4$, as well as all x-coordinates where that minimum is attained.

The short way: x^4 and $2x^2$ are both at least 0, so y cannot be less than 4. On the other hand, when $x = 0$, $y = 4$. So, 4 is the minimum value y can reach. As we already said, this happens when $x = 0$.

The long way: let $t = x^2$; then $y = t^2 + 2t + 4 = t^2 + 2t + 1 + 3 = (t+1)^2 + 3$. We can rewrite y in terms of x as $y = (x^2+1)^2 + 3$. To make this quantity as small as possible, we should make the quantity $(x^2 + 1)$ as close to zero as possible—which, since that quantity is positive, means we should minimize it. To minimize $x^2 + 1$, we should make x be as close to zero as possible. So, we choose $x = 0$; thus $x^2 + 1 = 0^2 + 1 = 1$; so $y = 1^2 + 3 = 4$.

Problem 4 *(10 points)*
Find all real solutions of the equation $2\cos\theta + \sin^2\theta = 1$.
Recall the Pythagorean Trig Identity $\sin^2\theta + \cos^2\theta = 1$. We'll use this to replace the $\sin^2\theta$ term with $1 - \cos^2\theta$:

$$\begin{aligned} 2\cos\theta + \sin^2\theta &= 1 \\ 2\cos\theta + 1 - cos^2\theta &= 1 \\ 2\cos\theta - cos^2\theta &= 0 \\ (\cos\theta)(2 - cos\theta) &= 0 \end{aligned}$$

In order for this to be true, we must have either $\cos\theta = 0$, or $2 - cos\theta = 0$. In the first case, recall that the points on the unit circle corresponding to a cosine of 0 are (0,1) and (0,-1): that is, points at odd multiples of $\pi/2$. In other words, $\theta = \pi/2 + k\pi$ for some integer k. In the second case, we have $cos\theta = 2$, which has no real solution. So, $\boxed{\theta = \pi/2 + k\pi}$.

Problem 5 *(10 points)*
Simplify, eliminating all exponents (and radical signs):

$$\begin{aligned}\log\left(\frac{10(x+2)^3(x-1)^3}{(x+4)^{1/2}}\right) &= \log\left(10(x+2)^3(x-1)^3\right) - \log\left((x+4)^{1/2}\right)\\ &= \log 10 + \log(x+2)^3 + \log(x-1)^3 - \log(x+4)^{1/2}\\ &= 1 + 3\log(x+2) + 3\log(x-1) - \frac{1}{2}\log(x+4)\end{aligned}$$

Problem 6 *(10 points)*
What is the reference angle of $5\pi/6$?
Since $5\pi/6$ is in the second quadrant, its reference angle is its supplement, $\pi/6$.

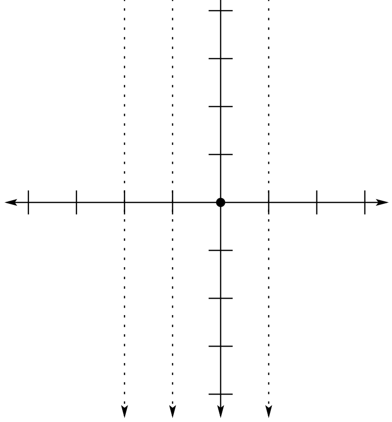
Evaluate the following:
$\sin\frac{5\pi}{6} = \pm\sin\frac{\pi}{6} = \pm\frac{1}{2}$. Because sine is positive in the second quadrant (where $5\pi/6$ lives), we decide that $\sin\frac{5\pi}{6} = \frac{1}{2}$.

$\cos\frac{5\pi}{6} = \pm\cos\frac{\pi}{6} = \pm\frac{\sqrt{3}}{2}$. Because cosine is negative in the second quadrant, we decide that $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$.
$\sec\frac{5\pi}{6} = \frac{1}{\cos(5\pi/6)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$.

Problem 7 *(10 points)*
Graph

$$y = \frac{x}{(x-1)^2(x+1)(x+2)}.$$

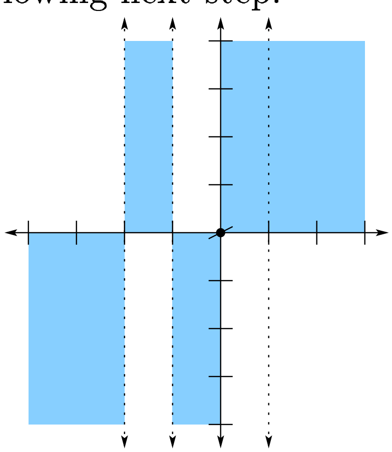
Identify vertical and horizontal asymptotes and any intercepts or other points of interest that you can.
The only root is at $x = 0$. The vertical asymptotes are $x = 1, -1, -2$. When we set $x = 0$, we find $y = 0$ too; thus, this is the y=intercept. Since the numerator has degree 1 and the denominator has degree 4, this function approaches the horizontal asymptote $y = 0$. Because the horizontal asymptote is the x-axis, a point of intersection between the function and its horizontal asymptote is also a root of the function; we have already found the only such point to be at the origin. So far, we have this picture:



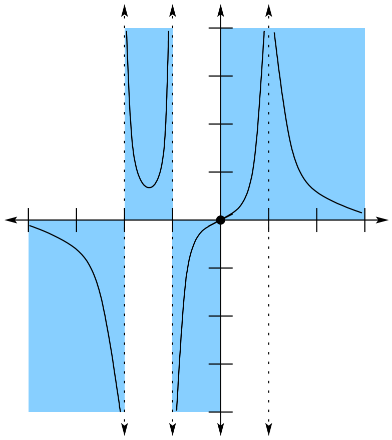
We could use the approximation technique near $x = 0, 1, -1, -2$ if we wanted. For instance:
If $x \approx 0$, $y \approx \frac{x}{(0-1)^2(0+1)(0+2)} = \frac{x}{2}$. So, this graph behaves like the line $y = \frac{x}{2}$ near the origin.
Another approach would be to make a sign chart. The four key numbers 0, 1, -1, -2 tell us where the function *could* switch sign. We break the number line into five intervals, splitting at those four numbers, and test the sign of y on each interval. It might go like this:

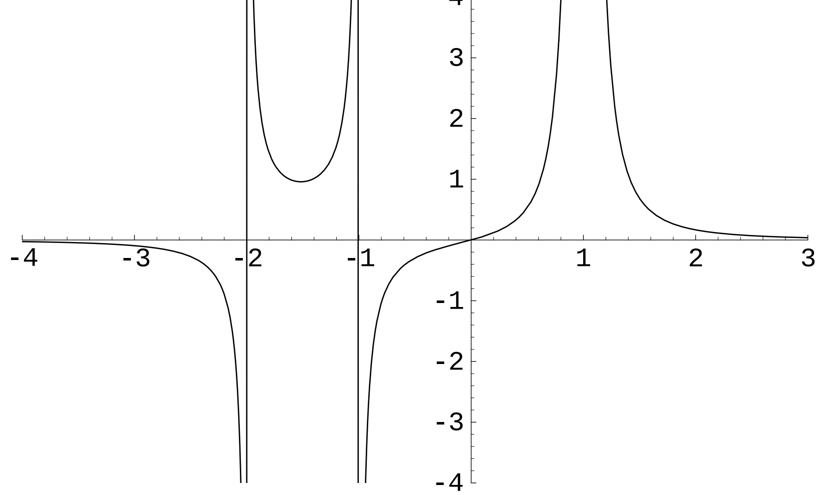
- If $x < -2$, then $y = \frac{-}{(-)^2(-)(-)} = -$.
- If $-2 < x < -1$, then $y = \frac{-}{(-)^2(-)(+)} = +$.
- If $-1 < x < 0$, then $y = \frac{-}{(-)^2(+)(+)} = -$.
- If $0 < x < 1$, then $y = \frac{+}{(-)^2(+)(+)} = +$.
- If $1 < x$, then $y = \frac{+}{(+)^2(+)(+)} = +$.

Pulling the approximation technique and the sign chart together, we get the following next step:



Finally, we draw the curve that seems to make the most sense. Compare to Mathematica's graph.





Problem 8 *(10 points)*
Given that $\pi/2 < \theta < \pi$ and $\tan\theta = -3/2$, find:

- $\sin\theta$
- $\cos\theta$
- $\sin 2\theta$

Let's start with $\tan\theta = -3/2$, and rearrange that until it becomes useful to us:

$$\begin{aligned} \tan\theta &= -3/2 \\ \frac{\sin\theta}{\cos\theta} &= -\frac{3}{2} \\ \sin\theta &= -\frac{3}{2}\cos\theta \end{aligned}$$

This is useful because we can drop this into the first Pythagorean Trig Identity, $\sin^2\theta + \cos^2\theta = 1$. We get:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \left(-\frac{3}{2}\cos\theta\right)^2 + \cos^2\theta &= 1 \\ \frac{9}{4}\cos^2\theta + \cos^2\theta &= 1 \\ \frac{13}{4}\cos^2\theta &= 1 \\ \cos^2\theta &= \frac{4}{13} \\ \cos\theta &= \pm\frac{2}{\sqrt{13}} \end{aligned}$$

Well, $\cos\theta$ can't be both positive AND negative, so we have to decide which it is. Recall that we were given $\pi/2 < \theta < \pi$, so θ is a second-quadrant angle. That means $\cos\theta$ is negative! So,

$$\cos\theta = -\frac{2}{\sqrt{13}}.$$

We can now say:

$$\sin\theta = -\frac{3}{2}\cos\theta = \left(-\frac{3}{2}\right)\left(-\frac{2}{\sqrt{13}}\right) = \frac{3}{\sqrt{13}}.$$

Finally, to find $\sin 2\theta$, we should use the double angle formula for sine:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{\sqrt{13}}\right)\left(-\frac{2}{\sqrt{13}}\right) = -\frac{12}{13}.$$

Problem 9 *(10 points)*
Find $\sin 15°$.
One method: Angle Difference. We notice that $15° = 45° - 30°$. Now we have

$$\begin{aligned}\sin 15° &= \sin(45° - 30°)\\ &= \sin 45° \cdot \cos 30° - \cos 45° \cdot \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Another method: Half Angle. We notice that $15° = 30°/2$. So, we find the sine of half of $30°$:

$$\sin 15° = \sin\frac{30°}{2} = \pm\sqrt{\frac{1-\cos 30°}{2}} = \pm\sqrt{\frac{1-\sqrt{3}/2}{2}}.$$

To decide whether to use the plus or the minus, note that $15°$ is in the first quadrant, so we use the positive root. Simplifying, we find that $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$.
Of course, our two answers look starkly different. One can show algebraically that these two numbers are actually equal. On your calculator they should be around .2588. Both answers are correct.

Problem 10 *(10 points)*
Find the domains of the following functions:
$f(x) = \sqrt{16 - x^2}$

We may take the square root of a nonnegative number. That is, we must have $16 - x^2 \geq 0$. You may notice that the graph of $y = 16 - x^2$ is a downward-opening parabola with roots at ± 4; thus, $y \geq 0$ when $-4 \leq x \leq 4$.

A more algebraic approach would be to factor the left hand side of the inequality: $(4 - x)(4 + x) \geq 0$. The key numbers for the expression on the left hand side are $x = \pm 4$. By testing a point left of -4, a point between ± 4, and a point to the right of 4, we will decide that the inequality is true only on the middle of the three intervals, namely [-4,4]. This is the same domain we found earlier, but written in interval notation.

$g(x) = \log(1 - 2x)$
We cannot take the log of a negative number, so we require $1 - 2x > 0$. That is, $1 > 2x$, or $x < \frac{1}{2}$. In interval notation, the domain is $(-\infty, \frac{1}{2})$.

Problem 11 Many years ago, I sailed the Cartesian Sea with a crew of swarthy mathematicians. We set out one day from the origin, and headed straight for (30,10). Our maps show a big rock at (1,7)—big enough to sink a ship. How close did we get to the rock, and where were we at the time?

The problem essentially boils down to identifying the line that represents the path of our ship, and then finding the point on that line which is closest to the rock (1,7). The ship sails straight from (0,0) to (30,10), so its slope is $\frac{10-0}{30-0} = 1/3$. Since the line passes through the origin, its y-intercept is 0, so the equation of the line is $y = x/3$. Let us suppose that the point on this line which is nearest to (1,7) has coordinates $(x, x/3)$. (Here we've used the equation of the line already, so that we don't have to think about the variable y.)

One approach to solving this problem is to find the distance between $(x, x/3)$ and (1,7), and then finding the choice of x that minimizes that distance. It uses our usual method for minimizing quadratic expressions. I detailed this solution during the final review.

A second (and faster) solution is to realize that if we really have found the point $(x, x/3)$ nearest (1,7), then the line through those two points will be perpendicular to the path of the ship. The slope of a line perpendicular to the path of the ship must be -3 (the negative reciprocal of 1/3), and the slope of the line though $(x, x/3)$ and (1,7) is $\frac{\frac{x}{3}-7}{x-1}$. That is,

$$\frac{\frac{x}{3} - 7}{x - 1} = -3$$

$$
\begin{aligned}
\frac{x}{3} - 7 &= -3(x-1) \\
\frac{x}{3} - 7 &= -3x + 3 \\
x - 21 &= -9x + 9 \\
10x &= 30 \\
x &= 3 \\
y &= x/3 \quad = \quad 1
\end{aligned}
$$

That is, the point (3,1) is as close as we'll ever get to the rock: the distance is $D = \sqrt{(3-1)^2 + (1-7)^2} = \sqrt{4+36} = \sqrt{40}$ units.