Math 12 Midterm 1 Solutions, Fall 2005

Problem 1 *(11 points)*
Let P be the point (3,1) and let Q be the point (2,-3).

- How far is it from P to Q?

  The distance from P to Q is $\sqrt{(3-2)^2+(1+3)^2} = \sqrt{1+16} = \sqrt{17}$.

- Write an equation for the line through P and Q.

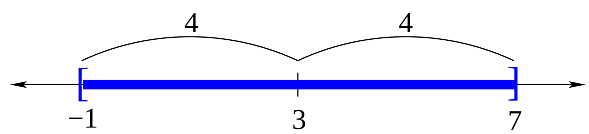
  The slope is $m = \frac{1+3}{3-2} = \frac{4}{1}$, and a point on it is (3,1), so an equation for the line (in point-slope form) is $y - 1 = 4(x-3)$.

- Now consider C, the circle centered at P that passes through Q. Write an equation for the line tangent to C at Q.

  The line tangent at Q is perpendicular to the line through Q and the center, P. Which means, the line whose equation we just wrote down is perpendicular to the line we're about to write down. Since the previous line had slope $m = 4$, the tangent line's slope will be the negative reciprocal of that, $m = -\frac{1}{4}$. We also know the line goes through $Q = (2, -3)$, so an equation for the tangent line (in point-slope form) is $y + 3 = -\frac{1}{4}(x-2)$. You could convert this into slope-intercept form if you like, but that is not necessary.

  Another approach to this problem requires a knowledge of vectors, so I include it here only for those who are interested—this is not a solution I expect to find on anybody's exam. The vector (directed distance) from Q to P is (1,4), because $(2,-3)+(1,4) = (3,1)$. Using those numbers (1 and 4)as coefficients in a standard form linear equation, we can decribe any line perpendicular to that vector. So, our line will have an equation of the form $1x + 4y = c$. But what's c? We figure this out by plugging Q in for (x, y), because Q is on the line we want (so Q should satisfy the equation). This gives us $1(2)+4(-3) = c$, or $c = -10$. So, the tangent line has standard form equation $1x + 4y = -10$.

Problem 2 *(10 points)*
Express the inequality $|x-3| \leq 4$ as a sentence involving distance.
"The distance from x to 3 is at most 4."
Draw a number line. Shade the x-values that *your sentence* describes.
We should center up on $x = 3$ and shade out to 4 units in each direction, with *closed* endpoints (because of the "equal" in "equal to or less than"). Here's the picture:



Solve the inequality using an algebraic method; please show steps. Express your answer in interval notation.

$$|x-3| \leq 4$$
$$-4 \leq x-3 \leq 4$$
$$3-4 \leq x \leq 4+3$$
$$-1 \leq x \leq 7$$

So, we find that x belongs to the interval $[-1, 7]$.

Problem 3 *(12 points)*
Find the domain of each function. (You may assume in each case that it is the largest reasonable collection of real numbers.) Write each answer in interval notation.

- $f(x) = 7x^4 - 22x + 1.3$

  Domain: All real numbers.

- $g(x) = \sqrt{3-x} + \frac{2x+3}{x-1}$

  In order for the square root to make sense, the stuff underneath it must be nonnegative. This gives us $3 - x \geq 0$, or equivalently, $3 \geq x$. Also, in order for the fraction to make sense, we must not divide by zero; thus, $x - 1 \neq 0$. Equivalently, $x \neq 1$. Now we put them together: x can be any real number equal to or less than 3 except for 1.

  Domain: $(-\infty, 1) \cup (1, 3]$.

- $h(x) = \sqrt{x^2 - 3x + 2}$

  Here we just need that the stuff under the radical is nonnegative. That is:

  $$x^2 - 3x + 2 \geq 0$$
  $$(x-1)(x-2) \geq 0$$

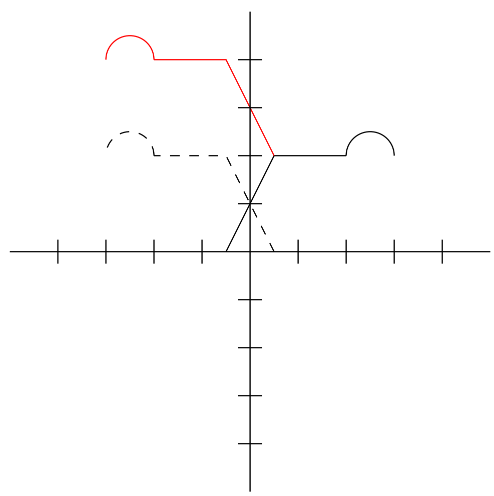
  Our key numbers are 1 and 2. Since $x^2 - 3x + 2$ describes an upward-opening parabola, we know that it will be negative between its roots and positive outside of them; that is, $x^2 - 3x + 2$ is nonnegative whenever $x \leq 1$ or $x \geq 2$.

  Domain: $(-\infty, 1] \cup [2, \infty)$.

Problem 4 *(12 points)*
Let $f(x) = 3x^2 + 2x - 1$. Assuming $h \neq 0$, simplify the following expression until it no longer has a denominator:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 + 2(x+h) - 1 - (3x^2 + 2x - 1)}{h} \\
&= \frac{3(x^2 + 2xh + h^2) + 2(x+h) - 1 - (3x^2 + 2x - 1)}{h} \\
&= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 1 - 3x^2 - 2x + 1}{h} \\
&= \frac{6xh + 3h^2 + 2h}{h} \\
&= \frac{h(6x + 3h + 2)}{h} \\
&= 6x + 3h + 2
\end{aligned}$$

Problem 5 *(10 points)*
Here is the graph of $y = f(x)$. On the same set of axes, sketch a graph of
$y = f(-x) + 2$.
The original function is shown in black, an intermediate step in dashed black, and the answer in red.



Problem 6 *(9 points)*
Consider the equation $x^3 + y^2 = 1$.

Is the graph of this equation symmetric with respect to the x-axis? $\boxed{\text{Yes}}$ / No

Is the graph of this equation symmetric with respect to the y-axis? Yes / $\boxed{\text{No}}$

Is the graph of this equation symmetric with respect to the origin? Yes / $\boxed{\text{No}}$

Problem 7 *(12 points)*
Let $f(x) = 2x - 3$, $g(x) = 4 - x$. Find:

- $(f + g)(x) = f(x) + g(x) = 2x - 3 + 4 - x = x + 1$
- $(f \circ g)(x) = f(g(x)) = 2(4 - x) - 3 = 8 - 2x - 3 = -2x + 5$
- $(g \circ f)(x) = g(f(x)) = 4 - (2x - 3) = 4 - 2x + 3 = -2x + 7$
- $f^{-1}(x)$

  We start with $y = 2x - 3$, switch x with y to get $x = 2y - 3$, and solve for y:

$$\begin{aligned} x &= 2y - 3 \\ x + 3 &= 2y \\ \frac{x+3}{2} &= y \\ y &= \frac{1}{2}(x + 3) \\ f^{-1}(x) &= \frac{1}{2}(x + 3) \end{aligned}$$

Problem 8 *(10 points)*
Find the average rate of change of $y = x^2 - 4x$ on the interval [-1,2].

We evaluate the function at the endpoints.
When $x = -1$, $y = (-1)^2 - 4(-1) = 1 + 4 = 5$.
When $x = 2$, $y = (2)^2 - 4(2) = 4 - 8 = -4$.
The average rate of change of the function is the slope of the line through these endpoints, (-1,5) and (2,-4). That is:

$$m = \frac{-4 - 5}{2 + 1} = \frac{-9}{3} = -3$$

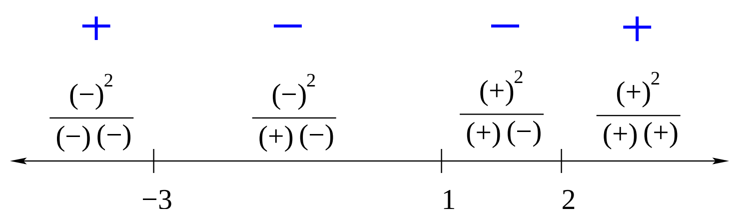
Problem 9 *(14 points)*
Solve the following inequality.

$$\frac{x^2 - 2x + 1}{x^2 + x - 6} \geq 0$$

We start by factoring top and bottom:

$$\frac{(x-1)^2}{(x+3)(x-2)} \geq 0.$$

The key numbers for this are 1, -3, and 2. In left-to-right order, those are -3, 1, 2. We break the number line at these key numbers and build the following sign chart:



The solution will now include any interval on which the function is positive, together with $x = 1$ (where the value is zero). However, it will *not* contain -3 or 2, where the function is undefined. So, the solution set is

$$(-\infty, -3) \cup \{1\} \cup (2, \infty)$$

Or, we could say that the inequality is true whenever $x < -3$, $x = 1$, or $x > 2$.

Extra credit: What recipe can be found in the Odds and Ends section of our course webpage? I'm not telling. But I will say that you can find the answer in the paper on Midpoints.