Math 12 Midterm 1 Solutions, Fall 2005

Problem 1 *(11 points)*
Let P be the point (2,4) and let Q be the point (-1,2).

- How far is it from P to Q?

  The distance from P to Q is $\sqrt{(2+1)^2+(4-2)^2} = \sqrt{9+4} = \sqrt{13}$.

- Write an equation for the line through P and Q.

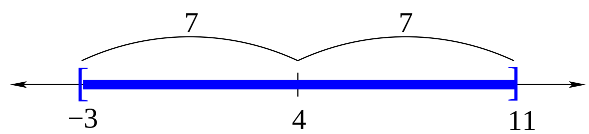
  The slope is $m = \frac{4-2}{2+1} = \frac{2}{3}$, and a point on it is (2,4), so an equation for the line (in point-slope form) is $y - 4 = \frac{2}{3}(x-2)$.

- Now consider C, the circle centered at P that passes through Q. Write an equation for the line tangent to C at Q.

  The line tangent at Q is perpendicular to the line through Q and the center, P. Which means, the line whose equation we just wrote down is perpendicular to the line we're about to write down. Since the previous line had slope $m = \frac{2}{3}$, the tangent line's slope will be the negative reciprocal of that, $m = -\frac{3}{2}$. We also know the line goes through $Q = (-1, 2)$, so an equation for the tangent line (in point-slope form) is $y - 2 = -\frac{3}{2}(x+1)$. You could convert this into slope-intercept form if you like, but that is not necessary.

  Another approach to this problem requires a knowledge of vectors, so I include it here only for those who are interested—this is not a solution I expect to find on anybody's exam. The vector (directed distance) from Q to P is (3,2), because $(-1,2)+(3,2) = (2,4)$. Using those numbers (3 and 2)as coefficients in a standard form linear equation, we can decribe any line perpendicular to that vector. So, our line will have an equation of the form $3x + 2y = c$. But what's c? We figure this out by plugging Q in for (x, y), because Q is on the line we want (so Q should satisfy the equation). This gives us $3(-1)+2(2) = c$, or $c = 1$. So, the tangent line has standard form equation $3x + 2y = 1$.

Problem 2 *(10 points)*
Express the inequality $|x - 4| \leq 7$ as a sentence involving distance.
"The distance from x to 4 is at most 7."
Draw a number line. Shade the x-values that *your sentence* describes.
We should center up on $x = 4$ and shade out to 7 units in each direction, with *closed* endpoints (because of the "equal" in "equal to or less than"). Here's the picture:



Solve the inequality using an algebraic method; please show steps. Express your answer in interval notation.

$$|x - 4| \leq 7$$
$$-7 \leq x - 4 \leq 7$$
$$4 - 7 \leq x \leq 7 + 4$$
$$-3 \leq x \leq 11$$

So, we find that x belongs to the interval $[-3, 11]$.

Problem 3 *(12 points)*
Find the domain of each function. (You may assume in each case that it is the largest reasonable collection of real numbers.) Write each answer in interval notation.

- $f(x) = x^5 - 47x^4 - 3.6$

  Domain: All real numbers.

- $g(x) = \sqrt{5-x} + \frac{2x+3}{x}$

  In order for the square root to make sense, the stuff underneath it must be nonnegative. This gives us $5 - x \geq 0$, or equivalently, $5 \geq x$. Also, in order for the fraction to make sense, we must not divide by zero; thus, $x \neq 0$. Now we put them together: x can be any real number equal to or less than 5 except for 0.

  Domain: $(-\infty, 0) \cup (0, 5]$.

- $h(x) = \sqrt{x^2 + x - 6}$

  Here we just need that the stuff under the radical is nonnegative. That is:

$$x^2 + x - 6 \geq 0$$
$$(x+3)(x-2) \geq 0$$

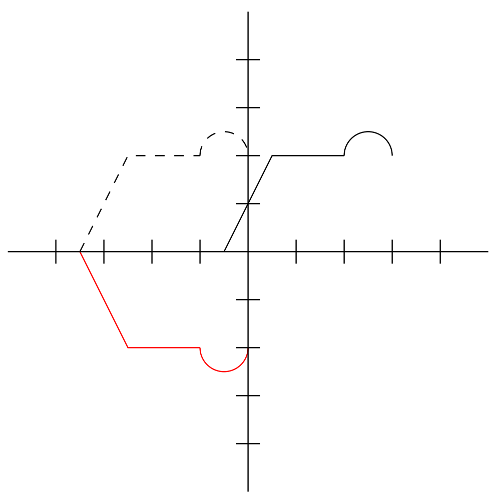
  Our key numbers are -3 and 2. Since $x^2 + x - 6$ describes an upward-opening parabola, we know that it will be negative between its roots and positive outside of them; that is, $x^2 + x - 6$ is nonnegative whenever $x \leq -3$ or $x \geq 2$.

  Domain: $(-\infty, -3] \cup [2, \infty)$.

Problem 4 *(12 points)*
Let $f(x) = 2x^2 + 3x + 2$. Assuming $h \neq 0$, simplify the following expression until it no longer has a denominator:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 + 3(x+h) + 2 - (2x^2 + 3x + 2)}{h} \\
&= \frac{2(x^2 + 2xh + h^2) + 3(x+h) + 2 - (2x^2 + 3x + 2)}{h} \\
&= \frac{2x^2 + 4xh + 2h^2 + 3x + 3h + 2 - 2x^2 - 3x - 2}{h} \\
&= \frac{4xh + 2h^2 + 3h}{h} \\
&= \frac{h(4x + 2h + 3)}{h} \\
&= 4x + 2h + 3
\end{aligned}$$

Problem 5 *(10 points)*
Here is the graph of $y = f(x)$. On the same set of axes, sketch a graph of
$y = -f(x+3)$.
The original function is shown in black, an intermediate step in dashed black, and the answer in red.



Problem 6 *(9 points)*
Consider the equation $x^2 + y^3 = 1$.

Is the graph of this equation symmetric with respect to the x-axis? Yes / $\boxed{\text{No}}$

Is the graph of this equation symmetric with respect to the y-axis? $\boxed{\text{Yes}}$ / No

Is the graph of this equation symmetric with respect to the origin? Yes / $\boxed{\text{No}}$

Problem 7 *(12 points)*
Let $f(x) = 3x - 2$, $g(x) = 5 - 2x$. Find:

- $(f + g)(x) = f(x) + g(x) = 3x - 2 + 5 - 2x = x + 3$
- $(f \circ g)(x) = f(g(x)) = 3(5 - 2x) - 2 = 15 - 6x - 2 = -6x + 13$
- $(g \circ f)(x) = g(f(x)) = 5 - 2(3x - 2) = 5 - 6x + 4 = -6x + 9$
- $f^{-1}(x)$

  We start with $y = 3x - 2$, switch x with y to get $x = 3y - 2$, and solve for y:

$$\begin{aligned} x &= 3y - 2 \\ x + 2 &= 3y \\ \frac{x+2}{3} &= y \\ y &= \frac{1}{3}(x + 2) \\ f^{-1}(x) &= \frac{1}{3}(x + 2) \end{aligned}$$

Problem 8 *(10 points)*
Find the average rate of change of $y = x^2 - 3x$ on the interval [1,4].

We evaluate the function at the endpoints.
When $x = 1$, $y = (1)^2 - 3(1) = 1 - 3 = -2$.
When $x = 4$, $y = (4)^2 - 3(4) = 16 - 12 = 4$.
The average rate of change of the function is the slope of the line through these endpoints, (1,-2) and (4,4). That is:

$$m = \frac{4+2}{4-1} = \frac{6}{3} = 2$$

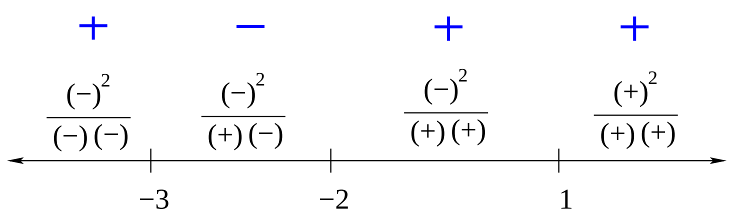
Problem 9 *(14 points)*
Solve the following inequality.

$$\frac{x^2 - 2x + 1}{x^2 + 5x + 6} \geq 0$$

We start by factoring top and bottom:

$$\frac{(x-1)^2}{(x+3)(x+2)} \geq 0.$$

The key numbers for this are 1, -3, and -2. In left-to-right order, those are -3, -2, 1. We break the number line at these key numbers and build the following sign chart:



The solution will now include any interval on which the function is positive, together with $x = 1$ (where the value is zero). However, it will *not* contain -3 or -2, where the function is undefined. So, the solution set is

$$(-\infty, -3) \cup (-2, \infty)$$

Or, we could say that the inequality is true whenever $x < -3$ or $x > -2$.

Extra credit: What recipe can be found in the Odds and Ends section of our course webpage? I'm not telling. But I will say that you can find the answer in the paper on Midpoints.