Math 12 Midterm 1 Solutions, Fall 2005

Problem 1 *(11 points)*
Let P be the point (-1,1) and let Q be the point (1,4).

- How far is it from P to Q?

  The distance from P to Q is $\sqrt{(1+1)^2+(4-1)^2} = \sqrt{4+9} = \sqrt{13}$.

- Write an equation for the line through P and Q.

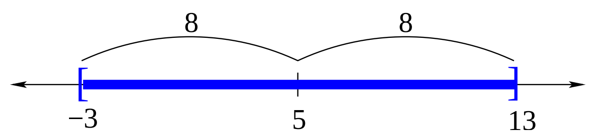
  The slope is $m = \frac{4-1}{1+1} = \frac{3}{2}$, and a point on it is (-1,1), so an equation for the line (in point-slope form) is $y - 1 = \frac{3}{2}(x+1)$.

- Now consider C, the circle centered at P that passes through Q. Write an equation for the line tangent to C at Q.

  The line tangent at Q is perpendicular to the line through Q and the center, P. Which means, the line whose equation we just wrote down is perpendicular to the line we're about to write down. Since the previous line had slope $m = \frac{3}{2}$, the tangent line's slope will be the negative reciprocal of that, $m = -\frac{2}{3}$. We also know the line goes through $Q = (1,4)$, so an equation for the tangent line (in point-slope form) is $y - 4 = -\frac{2}{3}(x-1)$. You could convert this into slope-intercept form if you like, but that is not necessary.

  Another approach to this problem requires a knowledge of vectors, so I include it here only for those who are interested—this is not a solution I expect to find on anybody's exam. The vector (directed distance) from P to Q is (2,3), because $(-1,1)+(2,3) = (1,4)$. Using those numbers (2 and 3) as coefficients in a standard form linear equation, we can decribe any line perpendicular to that vector. So, our line will have an equation of the form $2x+3y=c$. But what's c? We figure this out by plugging Q in for (x,y), because Q is on the line we want (so Q should satisfy the equation). This gives us $2(1)+3(4)=c$, or $c = 14$. So, the tangent line has standard form equation $2x+3y=14$.

Problem 2 *(10 points)*
Express the inequality $|x - 5| \leq 8$ as a sentence involving distance.
"The distance from x to 5 is at most 8."
Draw a number line. Shade the x-values that *your sentence* describes.
We should center up on $x = 5$ and shade out to 8 units in each direction, with *closed* endpoints (because of the "equal" in "equal to or less than"). Here's the picture:



Solve the inequality using an algebraic method; please show steps. Express your answer in interval notation.

$$|x - 5| \leq 8$$
$$-8 \leq x - 5 \leq 8$$
$$5 - 8 \leq x \leq 8 + 5$$
$$-3 \leq x \leq 13$$

So, we find that x belongs to the interval $[-3, 13]$.

Problem 3 *(12 points)*
Find the domain of each function. (You may assume in each case that it is the largest reasonable collection of real numbers.) Write each answer in interval notation.

- $f(x) = 11x^3 - 12.7x^2 - 4$

  Domain: All real numbers.

- $g(x) = \sqrt{2-x} + \frac{2x+3}{x+1}$

  In order for the square root to make sense, the stuff underneath it must be nonnegative. This gives us $2 - x \geq 0$, or equivalently, $2 \geq x$. Also, in order for the fraction to make sense, we must not divide by zero; thus, $x + 1 \neq 0$. Equivalently, $x \neq -1$. Now we put them together: x can be any real number equal to or less than 2 except for -1.

  Domain: $(-\infty, -1) \cup (-1, 2]$.

- $h(x) = \sqrt{x^2 - x - 2}$

  Here we just need that the stuff under the radical is nonnegative. That is:

  $$x^2 - x - 2 \geq 0$$
  $$(x+1)(x-2) \geq 0$$

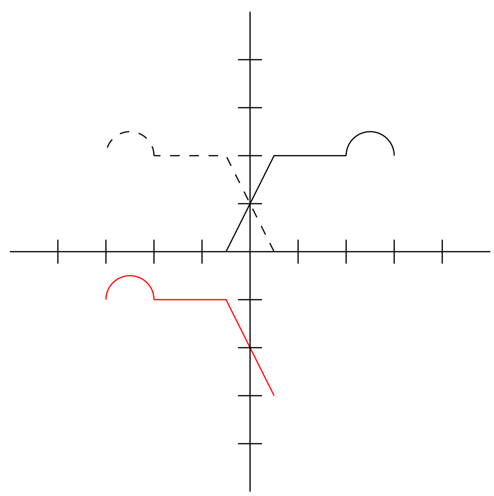
  Our key numbers are -1 and 2. Since $x^2 - x - 2$ describes an upward-opening parabola, we know that it will be negative between its roots and positive outside of them; that is, $x^2 - x - 2$ is nonnegative whenever $x \leq -1$ or $x \geq 2$.

  Domain: $(-\infty, -1] \cup [2, \infty)$.

Problem 4 *(12 points)*
Let $f(x) = 2x^2 - 4x + 1$. Assuming $h \neq 0$, simplify the following expression until it no longer has a denominator:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 - 4(x+h) + 1 - (2x^2 - 4x + 1)}{h} \\
&= \frac{2(x^2 + 2xh + h^2) - 4(x+h) + 1 - (2x^2 - 4x + 1)}{h} \\
&= \frac{2x^2 + 4xh + 2h^2 - 4x - 4h + 1 - 2x^2 + 4x - 1}{h} \\
&= \frac{4xh + 2h^2 - 4h}{h} \\
&= \frac{h(4x + 2h - 4)}{h} \\
&= 4x + 2h - 4
\end{aligned}$$

Problem 5 *(10 points)*
Here is the graph of $y = f(x)$. On the same set of axes, sketch a graph of $y = f(-x) - 3$.
The original function is shown in black, an intermediate step in dashed black, and the answer in red.



Problem 6 *(9 points)*
Consider the equation $xy^3 = 1$.

Is the graph of this equation symmetric with respect to the x-axis? Yes / $\boxed{\text{No}}$

Is the graph of this equation symmetric with respect to the y-axis? Yes / $\boxed{\text{No}}$

Is the graph of this equation symmetric with respect to the origin? $\boxed{\text{Yes}}$ / No

Problem 7 *(12 points)*
Let $f(x) = 2x + 5$, $g(x) = 2 - 3x$. Find:

- $(f + g)(x) = f(x) + g(x) = 2x + 5 + 2 - 3x = 7 - x$
- $(f \circ g)(x) = f(g(x)) = 2(2 - 3x) + 5 = 4 - 6x + 5 = -6x + 9$
- $(g \circ f)(x) = g(f(x)) = 2 - 3(2x + 5) = 2 - 6x - 15 = -6x - 13$
- $f^{-1}(x)$

  We start with $y = 2x + 5$, switch x with y to get $x = 2y + 5$, and solve for y:

$$\begin{aligned} x &= 2y + 5 \\ x - 5 &= 2y \\ \frac{x-5}{2} &= y \\ y &= \frac{1}{2}(x - 5) \\ f^{-1}(x) &= \frac{1}{2}(x - 5) \end{aligned}$$

Problem 8 *(10 points)*
Find the average rate of change of $y = x^2 - 2x$ on the interval [-1,3].

We evaluate the function at the endpoints.
When $x = -1$, $y = (-1)^2 - 2(-1) = 1 + 2 = 3$.
When $x = 3$, $y = (3)^2 - 2(3) = 9 - 6 = 3$.
The average rate of change of the function is the slope of the line through these endpoints, (-1,3) and (3,3). That is:

$$m = \frac{3-3}{3+1} = \frac{0}{4} = 0$$

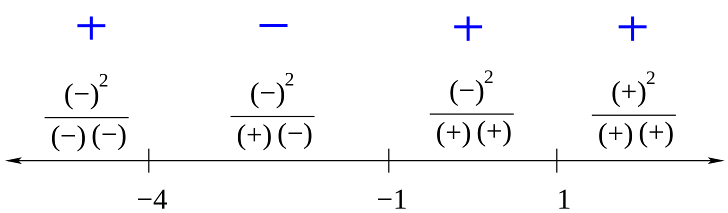
Problem 9 *(14 points)*
Solve the following inequality.

$$\frac{x^2 - 2x + 1}{x^2 + 5x + 4} \geq 0$$

We start by factoring top and bottom:

$$\frac{(x-1)^2}{(x+4)(x+1)} \geq 0.$$

The key numbers for this are 1, -4, and -1. In left-to-right order, those are -4, -1, 1. We break the number line at these key numbers and build the following sign chart:



The solution will now include any interval on which the function is positive, together with $x = 1$ (where the value is zero). However, it will *not* contain -4 or -1, where the function is undefined. So, the solution set is

$$(-\infty, -4) \cup (-1, \infty)$$

Or, we could say that the inequality is true whenever $x < -4$ or $x > -1$.

Extra credit: What recipe can be found in the Odds and Ends section of our course webpage? I'm not telling. But I will say that you can find the answer in the paper on Midpoints.