Math 12 Practice Final, Winter Quarter 2004

Problem 1 Find the domain:

- $f(x) = 4 - x$

  This is a polynomial, so its domain is $(-\infty, \infty)$.

- $f(x) = \sqrt{4 - x}$

  For the square root to make sense, we must have $4 - x \geq 0$. We rearrange this as $4 \geq x$, or $x \leq 4$. In interval notation, that's $(-\infty, 4]$.

- $f(x) = \log(4 - x)$

  This function is slightly more picky, in that $4 - x$ cannot actually equal zero. So, we have $4 - x > 0$, which means $x < 4$. That's also known as $(-\infty, 4)$.

- $f(x) = \sqrt{\log(4 - x)}$

  Ah, the challenge. To keep the square root *and* the log happy, we must satisfy two conditions simultaneously: $4 - x > 0$ (in order for the log to make sense), and $\log(4 - x) \geq 0$ (in order for the square root to make sense). The first for these conditions is already familiar: if $4 - x > 0$, then $4 > x$ and $\boxed{x < 4}$. The second condition is more elaborate.

$$\begin{aligned} \log(4-x) &\geq 0 \\ 10^{\log(4-x)} &\geq 10^0 \\ 4 - x &\geq 1 \\ 4 &\geq x + 1 \\ 3 &\geq x \\ x &\leq 3 \end{aligned}$$

  It turns out that any solution to the inequality $x \leq 3$ also satisfies the earlier one, so that's our domain. (If you draw both intervals on the same number line, you will see that one is contained in the other.) We may also write the domain as $(-\infty, 3]$.

Find the range:

- $f(x) = 3x + 2$

  An odd-degree polynomial has range "All real numbers", or $(-\infty, \infty)$. (Oh yeah—I almost forgot to mention, this is one of those.)

- $f(x) = (3x+2)^2 + 1$

  We may rewrite this as:

$$\begin{aligned} f(x) &= (3x+2)^2 + 1 \\ f(x) &= (3x+2)(3x+2) + 1 \\ f(x) &= 3(x+\frac{2}{3})(3x+2) + 1 \\ f(x) &= 3(x+\frac{2}{3})3(x+\frac{2}{3}) + 1 \\ f(x) &= 9(x+\frac{2}{3})^2 + 1 \end{aligned}$$

  As we see, this is an upward-opening parabola with vertex $(-2/3, 1)$. The minimum y-value is 1, and it assumes all values larger than that; thus, the range is $[1, \infty)$. You might have gotten the same answer faster by looking at the original equation, $f(x) = (3x+2)^2 + 1$, and noting the following: if $3x + 2 = 0$, then $y = 0^2 + 1 = 1$. If $3x + 2$ is positive OR negative, y is bigger than one. While this is slightly less than the whole story, it should at least give us reason to suspect the range is $[1, \infty)$.

- $f(x) = \cos x$

  Range: $[-1, 1]$.

- $f(x) = \frac{4-x}{3x+2}$

  One way to do this is to find the domain of the inverse. We swap x for y and y for x, solve for y, and find the domain:

$$\begin{aligned} y &= \frac{4-x}{3x+2} \\ x &= \frac{4-y}{3y+2} \\ (3y+2)x &= 4-y \\ 3xy + 2x &= 4-y \\ 3xy + y &= 4-2x \\ (3x+1)y &= 4-2x \\ y &= \frac{4-2x}{3x+1} \end{aligned}$$

  Since we can't have a zero denominator, this function has as its domain: all real numbers except $-1/3$. Thus, the range of the original function is all real numbers except $-1/3$. (You may recognize $y = -1/3$ as the equation of the horizontal asymptote.)

Problem 2 What is the average rate of change of $y = \sqrt{x}$ on the interval [1,4]?

$$\frac{\sqrt{4}-\sqrt{1}}{4-1} = \frac{2-1}{4-1} = \frac{1}{3}$$

What is the average rate of change of $y = 2^x$ on the interval [1,4]?

$$\frac{2^4-2^1}{4-1} = \frac{16-2}{3} = \frac{14}{3}$$

What is the average rate of change of $y = \sqrt{x} + 2^x$ on the interval [1,4]?

$$\frac{(\sqrt{4}+2^4)-(\sqrt{1}+2^1)}{4-1} = \frac{(2+16)-(1+2)}{3} = \frac{15}{3} = 5$$

Problem 3 Rational function–see midterm 2 solutions.

Problem 4 Solve:

- $t^2 + 3t - 4 = 0$

  Since $(t+4)(t-1) = 0$, we have $t = -4$ or $t = 1$.

- $3\cos A - \sin^2 A = 3$

  Recall the pythagorean identity $\sin^2 A + \cos^2 A = 1$, or $\sin^2 A = 1 - \cos^2 A$. Using this to substitute for $\sin^2 A$, we get:

$$\begin{aligned} 3\cos A - \sin^2 A &= 3 \\ 3\cos A - (1-\cos^2 A) &= 3 \\ \cos^2 A + 3\cos A - 1 &= 3 \\ \cos^2 A + 3\cos A - 4 &= 0 \end{aligned}$$

  You may not need to substitute here, but I think that's the easiest way to move forward: let $t = \cos A$. We get $t^2 + 3t - 4 = 0$, which we already solved; the solutions for t are -4 and 1. That is, we have $\cos A = -4$ or $\cos A = 1$. The former of these is impossible , because $\cos A$ is between 1 and -1 (as we decided on an earlier problem). But the latter is possible: $\cos A = 1$ whenever A is an even multiple of π. That is, our solution (in A) is $A = 2k\pi$ for some integer k.

- $5^{4x} + 3(5^{2x}) - 4 = 0$

  Similar stunt. Let $t = 5^{2x}$. Now $t^2 = 5^{4x}$, so the above formula reduces to $t^2 + 3t - 4 = 0$. Again, we've already solved this—how convenient! We have $t = -4$ or 1, so $5^{2x} = -4$ or $5^{2x} = 1$. The former is impossible, as 5^{2x} is always positive; the latter is possible if $2x = 0$. That's because $5^0 = 1$. Not impressed? Take the $\log_5$ of both sides of the equation $5^{2x} = 1$. Anyway, the only choice for x is 0.

Problem 5 Write as a combination of several logarithms, with no exponents:

$$\log_3 \left(\frac{\sqrt{(x-4)^3}(x^4)}{9(x+2)(x+1)^5} \right)$$

This is equivalent to

$$\frac{3}{2} \log_3(x-4) + 4\log_3 x - \log_3 9 - \log_3(x+2) - 5\log_3(x+1).$$

We could simplify slightly by using the fact that $\log_3 9 = 2$.

Problem 6 Find $\sin 13\pi/6$, $\cos(-3\pi/4)$.
First, we should find $13\pi/6$ on the unit circle. It's the same as $12\pi/6 + \pi/6 = 2\pi + \pi/6$, which for our purposes, is the same as just $\pi/6$. This angle is familiar; its sine is 1/2. So, $\sin(13\pi/6) = 1/2$.
A little trickier is $\cos(-3\pi/4)$. The reference angle of $-3\pi/4$ is $\pi/4$. Note that $\cos \pi/4 = \frac{\sqrt{2}}{2}$. Since $-3\pi/2$ is in the third quadrant (where cosine is negative), we have $\cos(-3\pi/4) = -\frac{\sqrt{2}}{2}$.

Problem 7 Given that $\pi < \theta < 3\pi/2$ and $\tan\theta = 1/4$, find $\sin(2\theta)$.

The double angle formula tells us that $\sin(2\theta) = 2\sin\theta\cos\theta$. So we should find THOSE values. We draw a triangle in the third quadrant; its vertices are the origin, (-4,0), and (-4,-1). (It is built so that the opposite side has length 1 and the adjacent side has length 4.) Now the hypotenuse has length $\sqrt{1^2 + 4^2} = \sqrt{17}$. That is, $\sin\theta = \pm\frac{1}{\sqrt{17}}$ and $\cos\theta = \pm\frac{4}{\sqrt{17}}$. In this case, both values are negative, since we're in the third quadrant. So, $\sin(2\theta) = 2\left(-\frac{1}{\sqrt{17}}\right)\left(-\frac{4}{\sqrt{17}}\right) = \frac{8}{17}$.

Problem 8 Find $\cos 15°$.

We use the half-angle identity:

$$\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

with $\theta = 30°$. In this case, it says

$$\cos 15° = \pm\sqrt{\frac{1+\cos 30°}{2}} = \pm\sqrt{\frac{1+\sqrt{3}/2}{2}} = \pm\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}.$$

Since 15° is in the first quadrant, we choose the positive option.

Problem 9 Find the equation of the line through the origin, parallel to the line through (1,2) and (2,5).

The line through (1,2) and (2,5) has slope $(5-2)/(2-1) = 3$. We want a line parallel to this, so its slope should also be 3. The line through the origin with slope 3 is $y = 3x$. (For the final, be prepared to use slope intercept, point slope, or standard forms; know about aprallel and perpendicular lines.)