

Practice Midterm 2 for Math 12, Winter Quarter, 2004

This study guide is a *fairly* thorough list of the ideas I want you to master, but it doesn't cover every last thing. I'll post some sample problems if I find the time, but you'll find your book is a great source of them.

Midterm 2 will cover sections 4.2, 4.4-4.7, and 5.1-5.5.

Key ideas from Chapter 4:

- Complete the square to convert from the form $y = ax^2 + bx + c$ to the form $y = a(x - h)^2 + k$.
- Know how to find the vertex and axis of symmetry of a parabola from its equation.
- Know what happens to the graph of $y = ax^2 + bx + c$ when the values of a or c are changed.
- Know what happens to the graph of $y = a(x - h)^2 + k$ when the values of a , h , or k are changed.
- Be able to find the minimum or maximum value of a quadratic function, or a function closely related to a quadratic (as in 4.2 examples 5-7).
- Be able to extract relevant information from a word problem, represent it in a diagram and/or system of equations, solve the word problem mathematically, and interpret your solution in the context of the original problem.
- Be familiar with various properties of polynomials in general, and of polynomials with a specified degree. In particular, you should be able to graph them and recognize their graphs, including in your analysis such things as:
 - Domain
 - Range
 - Roots
 - Continuity and “smoothness” (i.e., no cusps)
 - End behavior (i.e., how does a polynomial behave when x is a large positive or large negative number?)
 - Symmetry
 - Regions where the function is positive or negative
 - Behavior near roots
- Be able to graph rational functions by hand, including in your analysis such things as:

- Domain
- Range
- Roots (aka x -intercepts)
- Vertical asymptotes
- y -intercept, if any
- Horizontal or slant asymptote (see note below)
- Where (if at all) does the graph cross its non-vertical asymptote?
- Use of the approximation technique (as explained in 4.6 Example 4) to describe behavior near a root or vertical asymptote
- Regions where the function is positive or negative

Note: If a rational function has a smaller degree upstairs than downstairs, then it has the line $y = 0$ as a horizontal asymptote. If the degrees are equal, the graph will have a horizontal asymptote, but not at $y = 0$. If the degree of the numerator exceeds the degree of the denominator by one, then there will be a slant asymptote—this is discussed at various points in the book, as well as in the example I linked from Assignment 13. Problem 9 on p.261 shows what happens when the numerator beats the denominator by a wider margin, but rational functions such as these will not be on the test. Well, technically, any polynomial is a rational function, and I do plan to test you on polynomials of degree higher than one...but this is the only exception.

Key ideas from Chapter 5:

- For any positive number b other than 1, the equation $f(x) = b^x$ defines an exponential function with the following *analytical* properties. (Look at Figs.1 and 2b on p.274 as you read these.)
 - It is defined for all real numbers x , but $f(x)$ is always positive—that is to say, its domain is $(-\infty, \infty)$, while its range is $(0, \infty)$.
 - It is continuous (there's just one piece)
 - It is smooth (no cusps)
 - It is one-to-one (thus, its inverse is also a function)
 - It has no turning points; it either increases for all x (if $b > 1$) or decreases for all x (if $0 < b < 1$). (Contrast this with the behavior of, say, $y = x^3 - x$.)
 - It has a horizontal asymptote which it approaches on one side, but not the other.

- Exponential functions also have several *algebraic* properties. I couldn't find these in the chapter(!), but I expect you to know them. The algebraic properties of logarithms (we'll get to those) come from these properties of exponents. If b is a positive number, then:

- 1. $b^1 = b$
- 2. $b^0 = 1$
- $b^r b^s = b^{r+s}$
- $b^r / b^s = b^{r-s}$ (an important special case: if $r = 0$, then we get $1/b^s = b^{-s}$)
- $(b^r)^s = b^{rs}$

Note: in the strange case $b = 1$, these rules remain *true*, but not very *useful*. Check it out.

- Be able to graph $y = A + Cb^x$ for various choices of A , b , and C . See Fig 2, p.274 and Fig 3, p.275.
- Know some things about e . Read Rudin's quote at the beginning of 5.2.
- Be able to solve equations that involve exponents. When the bases involved are powers of a common base, such as in $4^x = 8^{2x-3}$ (here the 4 and the 8 are powers of 2), you can often get by without using logs of any kind. Sometimes you need to use logarithms. Know both methods.
- Know that the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ are inverses of each other (see p.155) for $b \neq 1$. Aside: if $b = 1$, then f isn't one-to-one (so it doesn't have an inverse function) and g doesn't make any sense. (Can you solve $1^x = 2$?)
- This relationship between b^x and $\log_b(x)$ shows up in graphs as well: see Fig 2, p.291; Fig 5a, p.293; Fig 6a, p.294.
- Be able to convert equations between exponential form and logarithmic form.
- By using the properties of exponential functions, together with the fact that logarithms are their inverses, we find several properties of logarithms. If b is a positive number other than 1, then the function $f(x) = \log_b(x)$ has these *analytical* properties, which are especially useful for graphing (compare these to the analytical properties of exponential functions):
 - It is defined only for positive real numbers x , but $f(x)$ can be any real number—that is to say, its domain is $(0, \infty)$, while its range is $(-\infty, \infty)$.
 - It is continuous (there's just one piece)

- It is smooth (no cusps)
 - It is one-to-one (What is its inverse?)
 - It has no turning points; it either increases for all x in its domain (if $b > 1$) or decreases for all x in its domain (if $0 < b < 1$). You won't be held accountable for graphs of $y = \log_b(x)$ for $b < 1$.
 - It has a vertical asymptote which it approaches on one side, but not the other.
- In addition to these analytical properties, there are also some (*very* useful) algebraic properties (compare to the list for exponential functions). For any positive b other than 1:
 - 1. $\log_b b = 1$
 - 2. $\log_b 1 = 0$
 - $\log_b(PQ) = \log_b P + \log_b Q$ “The log of a product is the sum of the logs.”
 - $\log_b(P/Q) = \log_b P - \log_b Q$ “The log of a quotient is the log of the top minus the log of the bottom.” An important special case: if $P = 1$, then we get $\log_b 1/Q = -\log_b Q$.
 - $\log_b(P^n) = n \log_b P$
 - $b^{\log_b P} = P$ (for $P > 0$) and $\log_b b^P = P$ (for all P), which restate the fact that logs and exponentials are inverses.

Enrichment tip: compare the “log of a product...” rule to the exponential function rule, $b^r b^s = b^{r+s}$. We might describe this latter rule as saying something like “the product of two exponentials is the exponential of the sum.” There is a similar rule-by-rule comparison, except for the last one (which really belongs to both lists).

- Understand the contents of the Errors To Avoid boxes on pp. 292, 306.
- Using various logarithm rules, be able to manipulate expressions involving logs: simplify them (as in 5.3 #13, 14, 27, 28 and 5.4 #1-10); break complicated terms into many simple terms (as in 5.4 #20-26); combine several terms into a single term (as in 5.4 #11-19). Well, *all* the exercises in 5.4 are pretty relevant.
- Be able to graph functions involving logarithmic or exponential terms, including in your analysis such things as:
 - Domain
 - Range
 - Roots (aka x -intercepts), if any

- y -intercept, if any
 - Horizontal or vertical asymptote
 - End behavior (i.e., how does the function behave very close to the asymptote, or very far from it?)
- Be able to solve equations and inequalities involving logarithmic and/or exponential terms, being mindful of extraneous roots and domain considerations. The examples of 5.5 (especially examples 4-9) illustrate this kind of problem.