Math 12 Midterm 1 Spring 2004

SOLUTIONS

Problem 1 *(10 points)*
Which one of these lines is NOT parallel to either of the other two? Explain.
A. $6x - 3y = 1$
B. $y = 2x - 5$
C. The line through (0,4) and (5,0).

SOLUTION:
Lines are parallel if they have the same slope. So, let's find slopes.
A.

$$\begin{aligned} 6x - 3y &= 1 \\ -3y &= -6x + 1 \\ y &= \boxed{2}x - \frac{1}{3} \end{aligned}$$

Slope=2

B.

$$y = \boxed{2}x - 5$$

Slope=2

C.

$$\begin{aligned} m &= \frac{0-4}{5-0} \\ &= -\frac{4}{5} \end{aligned}$$

Slope=-4/5

So, lines A and B had slope 2; line C is the non-parallel line.

Problem 2 *(10 points)*
Given that $x > 9$, simplify $|x-3|+|5-x|$ into a form that does not involve absolute value.

SOLUTION:
Since $x > 9$, we can say that $x - 3$ is positive and $5 - x$ is negative. (We could say, more specifically, that $x - 3 > 6$ and $5 - x < -4$, but that's more detail than we need.) Since $x - 3$ is positive, absolute value leaves it alone: $|x - 3| = x - 3$. Since $5 - x$ is negative, absolute value changes its sign: $|5 - x| = -(5 - x)$. So, we have:

$$|x - 3| + |5 - x| = x - 3 - (5 - x) = x - 3 - 5 + x = 2x - 8.$$

Problem 3 *(9 points)*
Find the domain of each function. Write each answer in interval notation.

- $f(x) = x^2 + 3x$

  SOLUTION:

  This is a polynomial, so it is defined for all real numbers x.

  Domain: $(-\infty, \infty)$.

- $g(x) = \sqrt{x^2 - 9}$

  SOLUTION:

  We can't take the square root of a negative number, so we must have

  $$\begin{aligned} x^2 - 9 &\geq 0 \\ (x - 3)(x + 3) &\geq 0 \end{aligned}$$

  The key numbers for this inequality are 3 and -3. That is, the domain of this function will comprise one or more of the intervals $(-\infty, -3]$, $[-3, 3]$, $[3, \infty)$. We may test the left interval with, say, $x = -4$: then we have $x^2 - 9 = (-4)^2 - 9 = 16 - 9 = 7 \geq 0$, so the inequality is true. This interval is part of the domain. We may test the middle interval with $x = 0$, if you like: then we have $x^2 - 9 = -9$, which is less than zero; the inequality is false. The middle interval is not part of the domain. We may test the right interval with $x = 5$: then we have $x^2 - 9 = 5^2 - 9 = 25 - 9 = 16 \geq 0$. So, the inequality is true, and our domain includes the rightmost interval.

  Domain: $(-\infty, -3] \cup [3, \infty)$

- $h(x) = \frac{x+2}{(x-1)(x+3)}$

  SOLUTION:

  Division is defined except when the denominator is zero. We must avoid that. Our denominator is zero when $x = 1$ or $x = -3$, so our domain is all real numbers except for these two.

  Domain: $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

Problem 4 *(12 points)*
Let $f(x) = 2x^2 - x$. Simplify the following expression until it no longer has a denominator:

$$\frac{f(x+h) - f(x)}{h}$$

SOLUTION:
The hardest part of this problem is probably knowing what $f(x+h)$ is. Keep in mind that if you were trying to find $f(4)$, you would just plug in 4 for x; use a similar idea to find $f(x+h)$. We just replace the x in $f(x)$ with an $x+h$:

$$f(x+h) = 2(x+h)^2 - (x+h) = 2(x^2+2xh+h^2) - x - h = 2x^2 + 4xh + 2h^2 - x - h.$$

Now we're ready for the whole expression.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(2x^2 + 4xh + 2h^2 - x - h) - (2x^2 - x)}{h} \\
&= \frac{2x^2 + 4xh + 2h^2 - x - h - 2x^2 + x}{h} \\
&= \frac{4xh + 2h^2 - h}{h} \\
&= \frac{h(4x + 2h - 1)}{h} \\
&= 4x + 2h - 1
\end{aligned}$$

Problem 5 *(10 points)*
Exactly how far is it from (3,1) to (2,5)?

SOLUTION:
We can use the distance formula:

$$d = \sqrt{(3-2)^2 + (1-5)^2} = \sqrt{1+16} = \sqrt{17}.$$

The Pythagorean Theorem works just as well if you forget the distance formula. The points (3,1), (2,1), and (2,5) form a right triangle. Draw the triangle and mark the lengths of its sides.

Problem 6 *(9 points)*
Consider the equation $x^3 + y^2 = 1$. With or without drawing a graph, explain your answers to the following:

Is the graph of this equation symmetric with respect to the x-axis? $\boxed{\text{Yes}}$ / No

SOLUTION:
To algebraically represent reflection of a graph in the x-axis, we should replace y with $-y$. If we do this in the given equation, we obtain $x^3 + (-y)^2 = 1$, which simplifies down to the original equation; thus, reflecting the graph in the x-axis doesn't actually change anything. The graph is symmetric with respect to the x axis.

Is the graph of this equation symmetric with respect to the y-axis? Yes / $\boxed{\text{No}}$

SOLUTION:
To algebraically represent reflection of a graph in the y-axis, we should replace x with $-x$. If we do this in the given equation, we obtain $(-x)^3 + y^2 = 1$, which simplifies down to $-x^3 + y^2 = 1$. This is not equivalent to the original equation; thus, reflecting the graph in the y-axis makes a difference. The graph is NOT symmetric with respect to the y axis.

Is the graph of this equation symmetric with respect to the origin? Yes / $\boxed{\text{No}}$

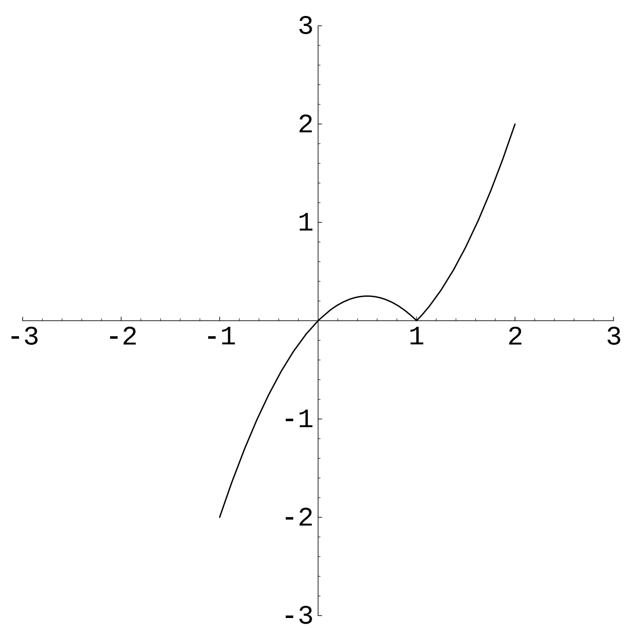
SOLUTION:
To algebraically represent reflection of a graph through the origin, we should replace x with $-x$, AND y with $-y$. If we do this in the given equation, we obtain $(-x)^3 + (-y)^2 = 1$, which simplifies down to $-x^3 + y^2 = 1$. Again, this is not equivalent to the original equation; thus, reflecting the graph through the origin changes something. The graph is NOT symmetric with respect to the origin.

Problem 7 *(10 points)*
What is the average rate of change of $y = x^2 + 2$ on the interval [-2,1]?

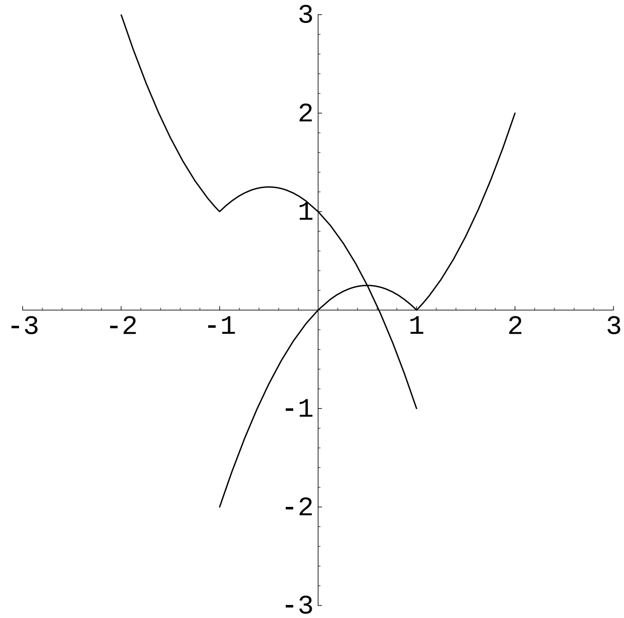
SOLUTION:
To find the average rate of change, we should calculate the endpoints (i.e., what the function is doing when $x = -2$ and when $x = 1$), and find the slope of the line through those endpoints. When $x = -2$, we find that $y = (-2)^2 + 2 = 4 + 2 = 6$. When $x = 1$, we find that $y = 1^2 + 2 = 3$. So, this graph goes through (-2,6) and (1,3). The average rate of change on the given interval is now

$$m = \frac{6-3}{-2-1} = \frac{3}{-3} = -1.$$

Problem 8 *(10 points)*
Here is a portion of the graph of $y = f(x)$. On the same set of axes, sketch a graph of $y = f(-x) + 1$.



SOLUTION:
The negative sign reflects the graph in the y-axis, and the $+1$ moves the graph up one unit. The two graphs, together, are:



Problem 9 *(10 points)*
Let $f(x) = x - 2$, $g(x) = 3x - 1$.
Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Are they equal?

SOLUTION:
$(f \circ g)(x) = f(g(x)) = f(3x - 1) = (3x - 1) - 2 = 3x - 1 - 2 = 3x - 3$.
$(g \circ f)(x) = g(f(x)) = g(x - 2) = 3(x - 2) - 1 = 3x - 6 - 1 = 3x - 7$.
They are not equal.

Problem 10 *(10 points)*
With f and g defined as in Problem 9, let $h = f/g$. Find $h^{-1}(x)$. Do you notice anything unusual about it?

SOLUTION:
To find $h^{-1}(x)$, we should follow these steps:

- Write $h(x)$ in terms of y
- Swap the roles of x and y
- Solve for y
- Replace y with $h^{-1}(x)$

So, this is how that looks:

$$
\begin{aligned}
h(x) &= \frac{f(x)}{g(x)} \\
h(x) &= \frac{x-2}{3x-1} && \text{Write in terms of } y \\
y &= \frac{x-2}{3x-1} && \text{Swap } x \text{ and } y \\
x &= \frac{y-2}{3y-1} && \text{Solve for } y \\
x(3y-1) &= y-2 \\
3xy - x &= y-2 \\
3xy - y &= x-2 \\
y(3x-1) &= x-2 \\
y &= \frac{x-2}{3x-1} && \text{Rewrite in terms of } h^{-1} \\
h^{-1}(x) &= \frac{x-2}{3x-1}
\end{aligned}
$$

The unusual thing we might notice is that, in this case, $h^{-1}(x)$ is the same function as $h(x)$. This does not happen very often.