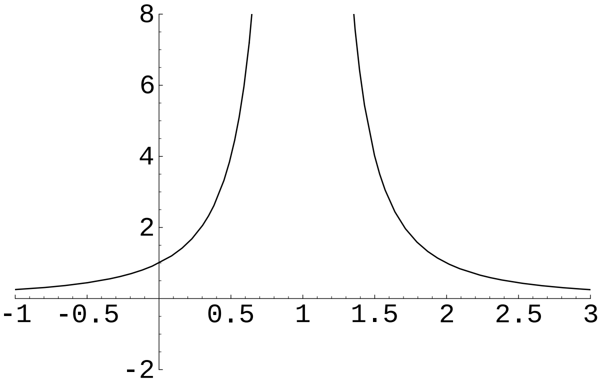
Problem 1 *(8 points)*

Match the following graphs to their functions, listed below.

A


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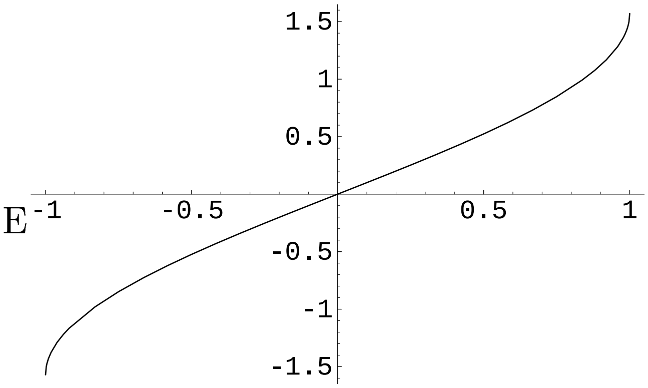
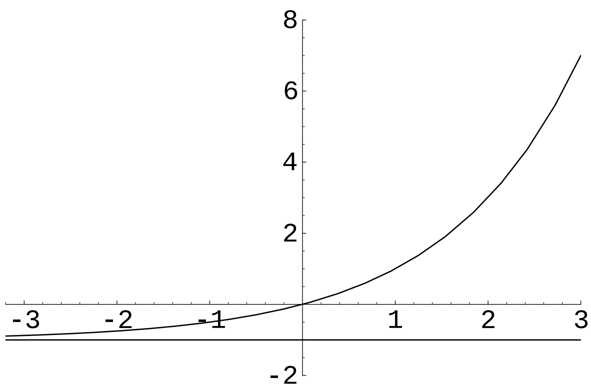


B


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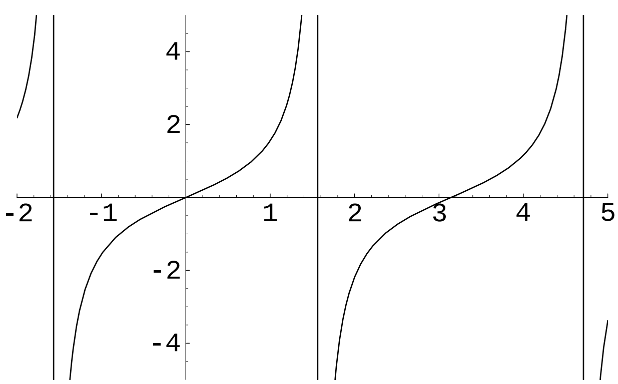

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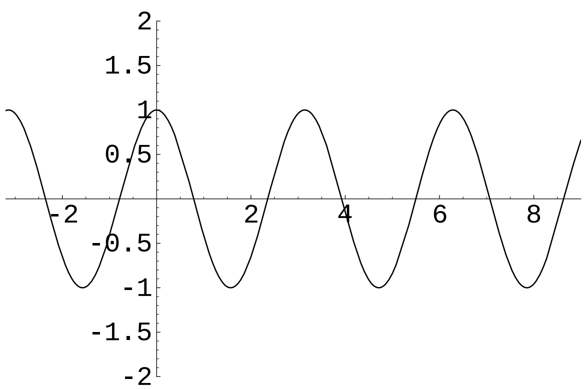

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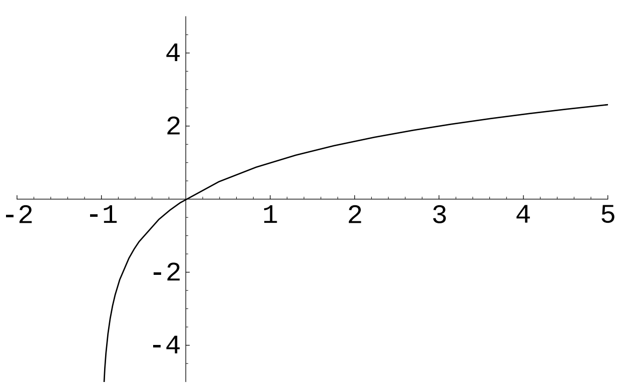
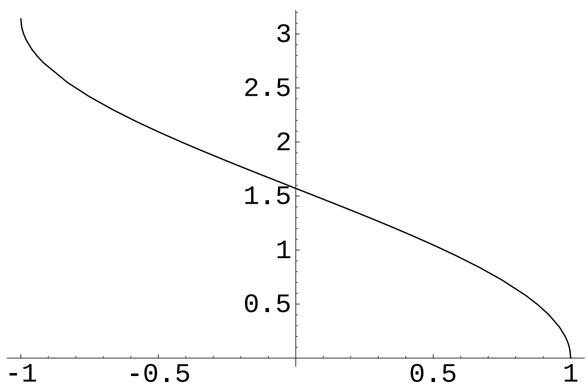
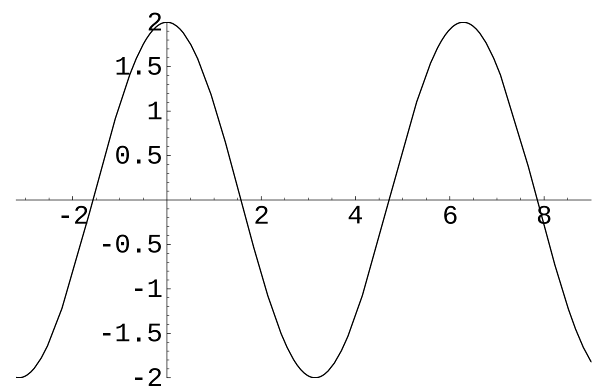


D


H


[C] $y = \cos 2x$ [H] $y = 2\cos x$ [B] $y = 2^x - 1$ [A] $y = \frac{1}{(x-1)^2}$

[G] $y = \log_2(x+1)$ [F] $y = \tan x$ [E] $y = \sin^{-1} x$ [D] $y = \cos^{-1} x$

Problem 2 *(8 points)*
Find the minimum value attained by the function $f(x) = x^4 - 6x^2 + 10$, as well as all x-coordinates where this minimum value occurs.

One way to solve this is to make a substitution, in order to bring out its quadratic nature. Let's use $T = x^2$. Now $x^4 - 6x^2 + 10 = T^2 - 6T + 10 = T^2 - 6T + 9 + 1 = (T - 3)^2 + 1$. As a function of T, this is a parabola with vertex (3,1). That means when $T = 3$, this function takes its minimum value of 1. (It's a minimum because the parabola opens upwards, to the vertex is at the bottom.) So, $\boxed{\text{the minimum value is 1}}$. While we know the T-coordinate where that happens, we haven't yet found the x-coordinate(s). To find x, we recall that $T = x^2$ and that the "best T" was $T = 3$. So, we solve $x^2 = 3$, getting $\boxed{x = \pm\sqrt{3}}$.

Problem 3 *(8 points)*
Use an angle sum, angle difference, or half angle identity to find $\sin 75°$. State the identity you use.

There are a few ways to go here. Probably the fastest is to notice that 75=45+30, where 45° and 30° are familiar angles. We use the Angle Sum Formula:

$$\begin{aligned}
\sin(75°) &= \sin(30° + 45°) \\
&= \sin(30°)\cos(45°) + \cos(30°)\sin(45°) \\
&= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
\sin(75°) &= \frac{\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

There are a few other correct ways this answer might appear.

Problem 4 *(8 points)*
Find the domains of the following functions. Express each domain in interval notation.

1. $f(x) = x - \sqrt{x}$

   The only restriction is that we can't take the square root of a negative number, so $x \geq 0$. That is, the domain is $[0, \infty)$.

2. $g(x) = \cos^{-1}(x)$

   The cosine of an angle is always between 1 and -1. So, when we're trying to find "the angle whose cosine is x" (which is the way of saying $\cos^{-1}(x)$ in English) the "x" has to be between -1 and 1. The domain is [-1,1].

3. $h(x) = \log(x) + \log(3 - x)$

   Logarithms are only happy when you give them positive numbers. The first one is only defined if $x > 0$; the second one requires $3 - x > 0$, or $x < 3$. So, the domain is (0,3).

4. $p(x) = 5x^3 + 4x^2 - 9$

   This is a polynomial, so its domain is all real numbers, $(-\infty, \infty)$.

Problem 5 *(10 points)*
Graph the following rational function. Locate and label all intercepts, asymptotes, and points where this function crosses an asymptote. You may use any valid method you know. Keep in mind that the easier it is to understand your steps, the easier it is to give partial credit.

$$y = \frac{(x+3)(x-1)}{(x+1)(x-2)}$$

The roots of the numerator, $x = -3$ and $x = 1$, tell us the x-intercepts of the graph. (If x is one of those numbers, then the whole function is "zero over nonzero", which is zero.)
The roots of the denominator, $x = -1$ and $x = 2$, tell us the vertical asymptotes of the graph. (If x is one of those numbers, then the whole function is "something over zero", which is undefined.)
To get the non-vertical asymptote, we compare the degrees of the upstairs and downstairs. We have a quadratic over a quadratic, so the degrees match—indicating a horizontal asymptote. (If you don't know where I got the word "quadratic" in that, multiply out the top and bottom of the fraction.) Since the leading coefficients are both 1 *and the degrees match*, we have a shortcut that says the horizontal asymptote is $y = \frac{1}{1} = 1$.
We should also find the y-intercept, by plugging in $x = 0$; we get $y = \frac{(3)(-1)}{(1)(-2)} = \frac{-3}{-2} = 1.5$. So, the point (0,1.5) is on our graph.
So far, we've found all intercepts (x and y) and asymptotes (vertical and nonvertical), so we should find all points where this function crosses an asymptote. It can't cross a vertical asymptote (because it's undefined there), but it might cross the horizontal asymptote. To see where, we set the function equal to the asymptote and solve for x:

$$\begin{aligned}
\frac{(x+3)(x-1)}{(x+1)(x-2)} &= 1 \\
(x+3)(x-1) &= (x+1)(x-2) \\
x^2 + 3x - x - 3 &= x^2 + x - 2x - 2 \\
x^2 + 2x - 3 &= x^2 - x - 2 \\
2x - 3 &= -x - 2 \\
3x &= 1 \\
x &= \frac{1}{3}
\end{aligned}$$

I don't have access to graphing software here...sad, I know. So, you'll have to sketch along with this description.

First, draw the asymptotes $x = -1$ and $x = 2$, which are vertical lines, and $y = 1$, which is a horizontal line. Mark the intercepts (-3,0), (1,0), and (0,1.5); also mark the asymptote crossing point $(\frac{1}{3}, 1)$. We know that y-coordinate is 1 because the point is on the horizontal asymptote. So far, our picture includes 3 vertical lines and 2 horizontal (counting the coordinate axes as well) and four points. The lines aren't part of the function's graph, but they do give us a framework: *the function cannot cross any of these lines except at the indicated points.*

We have three points in a row between the uprights (the vertical asymptotes), so we can connect them with a curve that rises on the left (towards the top end of $x = -1$) and falls on the right (towards the bottom end of $x = 2$).

Only one of our Four Points remains; it's at (-3,0). We need to draw a curve through this point—one that gets close to the horizontal asymptote at the left edge of the picture. So as we go right to left through the point, our curve should rise gently and approach, but not cross, the horizontal line $y = 1$. As we go left to right through (-3,0), our graph should fall faster and faster until it plummets off the bottom of the page somewhere near the bottom end of the vertical asymptote $x = -1$. (Don't cross that asymptote, either.)

Finally, we need to do something about the right side of our picture. Even knowing a single point over there would help; let's plug in 3 for x (because 3 is the smallest nice number bigger than 2, where $x = 2$ is that vertical asymptote). When $x = 3$, $y = (6 \cdot 2)/(4 \cdot 1) = 12/4 = 3$, so plot the point (3,3). It's in the upper right region. Now as we read right to left going through this point, our graph must approach one end of the vertical asymptote $x = 2$, without crossing that horizontal asymptote. So, it has to go up. It rises very quickly towards the top end of $x = 2$. As we go left to right, it approaches the right end of $y = 1$ (again, without crossing).

That's it for the graph...I hope you have access to a graphing calculator or something so you can compare a picture to these words.

Problem 6 *(10 points)*
Find all real solutions to the equation $2\cos\theta + \sin^2\theta = 1$. How many of these solutions lie in the interval $[0, 2\pi)$?

The strategy here is to eliminate $\sin^2\theta$. Since the only occurrance of the sine function is squared, we'll be able to use the pythagorean identity. We'll be left with an equation that is quadratic in cosine; we can factor this and finish the problem.

$$\begin{aligned} 2\cos\theta + \sin^2\theta &= 1 \\ 2\cos\theta &= 1 - \sin^2\theta \\ 2\cos\theta &= \cos^2\theta \\ 0 &= \cos^2\theta - 2\cos\theta \\ \cos^2\theta - 2\cos\theta &= 0 \\ (\cos\theta)(\cos\theta - 2) &= 0 \end{aligned}$$

Now we have $(\cos\theta)(\cos\theta - 2) = 0$. This is true if and only if one of the factors is zero; that is, either $\cos\theta = 0$ or $\cos\theta - 2 = 0$. This means $\cos\theta = 0$ or $\cos\theta = 2$. But it's impossible that $\cos\theta = 2$, so we're down to one equation: $\cos\theta = 0$.
To solve this using the unit circle, keep in mind that $\cos\theta$ is the x-coordinate of the point at angle θ. So, saying "$\cos\theta = 0$" is like saying "$x = 0$". Draw the line $x = 0$ (that's the y-axis) in your unit circle picture. Notice that it crosses the circle at the top, (0,1), and the bottom, (0,-1). The angles corresponding to these points are $\theta = \frac{\pi}{2}$ (plus any number of full turns) and $\theta = \frac{3\pi}{2}$ (plus any number of full turns). That is, our solution is $\boxed{\theta = \frac{\pi}{2} + 2\pi k,\ \ \frac{3\pi}{2} + 2\pi k}$. Only two of these ($\pi/2$ and $3\pi/2$) are in the interval $[0, 2\pi)$.

Problem 7 *(8 points)*
Let $\theta = \frac{7\pi}{6}$. Find:

1. $\sin\theta$

   $7\pi/6$ has reference angle $\pi/6$, so $\sin 7\pi/6 = \pm\sin\pi/6 = \pm\frac{1}{2}$. But $7\pi/6$ is in the third quadrant, where sine is negative (ASTC), so $\boxed{\sin\theta = -\frac{1}{2}}$.

2. $\cos\theta$

   Usig the same reference angle, we have $\cos 7\pi/6 = \pm\cos\pi/6 = \pm\frac{\sqrt{3}}{2}$. Cosine is also negative in the third quadrant, so $\boxed{\cos\theta = -\frac{\sqrt{3}}{2}}$.

3. $\tan\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

4. $\sec\theta$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}.$$

5. $\csc\theta$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{1}{2}} = -2$$

6. $\cot\theta$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{1/\sqrt{3}} = \sqrt{3}.$$

Problem 8 *(8 points)*
If $\cos\theta = 1/4$ and $-\frac{\pi}{2} < \theta < 0$, find:

1. $\sin\theta$

   We have that θ is in the fourth quadrant and $\cos\theta = 1/4$. By the pythagorean trig identity, we know that $\sin^2\theta + \cos^2\theta = 1$, so:

$$\begin{aligned}\sin^2\theta + \left(\frac{1}{4}\right)^2 &= 1\\ \sin^2\theta + \frac{1}{16} &= 1\\ \sin^2\theta &= \frac{15}{16}\\ \sin\theta &= \pm\frac{\sqrt{15}}{4}\end{aligned}$$

   We're in the fourth quadrant, where sine is negative, so $\boxed{\sin\theta = -\frac{\sqrt{15}}{4}}$.

2. $\tan\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{15}/4}{1/4} = -\sqrt{15}.$$

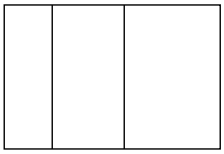
3. $\cos 2\theta$

   Here we use the double angle identity:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \left(\frac{1}{4}\right)^2 - \left(-\frac{\sqrt{15}}{4}\right)^2\\ &= \frac{1}{16} - \frac{15}{16}\\ &= \frac{-14}{16}\\ \cos 2\theta &= \frac{-7}{8}\end{aligned}$$

   As a quick check, we might notice that θ is in the fourth quadrant, so 2θ is in the third; thus, $\cos 2\theta$ ought to be negative. This checks out.

Problem 9 *(8 points)*
I want to fence a rectangular region, subdivided into three parts as shown (with all fences running north-south or east-west). Supposing I have 1000 feet of fence available and I enclose the maximum possible total area, what are the dimensions of the fenced region?



If we call the height of the diagram x and the width y, then the area is given by $A = xy$. In order to express this as a function of x, we'll need to use the constraint: the total amount of fence used is $4x + 2y = 1000$. So, $2y = 1000 - 4x$, giving us $y = 500 - 2x$. We replace the y in "$A = xy$" with this formula, giving us:

$$\begin{aligned} A &= xy \\ A &= x(500 - 2x) \\ A &= 2x(250 - x) \\ A &= -2x^2 + 500x \end{aligned}$$

The last two lines above are good for different shortcuts. Basically we'd like to find the vertex of the parabola given by either of those equations (it's the same parabola). We could apply the shortcut $x = -\frac{b}{2a}$ to the last line, getting $x = -\frac{500}{-4} = \frac{250}{2} = 125$. Or, we could note from the previous line that the parabola has roots $x = 0$ and $x = 250$, so its vertex lies halfway between—at $x = 125$. Either way, we get $x = 125$. Since $y = 500 - 2x$, we get $y = 250$ and $A = (125)(250) = 31250$. Oh, but the question just asked for the regions dimensions, which are 125 by 250.

Problem 10 *(8 points)*
Simplify $\ln(\log_4 28 - \log_4 7)$.

$$\ln(\log_4 28 - \log_4 7) = \ln\left[\log_4\left(\frac{28}{7}\right)\right] = \ln(\log_4 4) = \ln 1 = 0$$

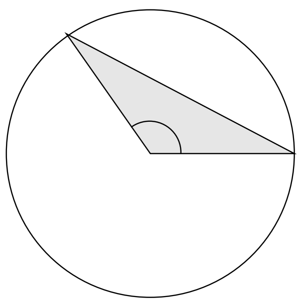
Problem 11 *(8 points)*
Solve for x:
$3(4 - e^x) \leq -2$

$$\begin{aligned} 3(4 - e^x) &\leq -2 \\ 12 - 3e^x &\leq -2 \\ 12 + 2 &\leq 3e^x \\ 14 &\leq 3e^x \\ \frac{14}{3} &\leq e^x \\ e^x &\geq \frac{14}{3} \\ x &\geq \ln\frac{14}{3} \end{aligned}$$

We can get to the last line from the one above it by either writing the inequality in logarithmic form, or by taking the natural log of both sides. If you used any other log, you could still get an answer; it might look like

$$x \geq \frac{\log(14/3)}{\log e}.$$

Problem 12 *(8 points)*
The circle below has radius 3, and the indicated angle measures 2.1 radians. What is the area of the triangle? (Your answer should be a formula involving numbers, but no variables—you don't have to simplify the result.)



The formula for the area of a triangle with sides a and b, and included angle θ, is $A = \frac{1}{2}ab\sin\theta$. In this case, both sides have length 3, so the area is

$$A = \frac{1}{2}(3^2)\sin 2.1 = \frac{9}{2}\sin 2.1.$$

And that's how we leave it.