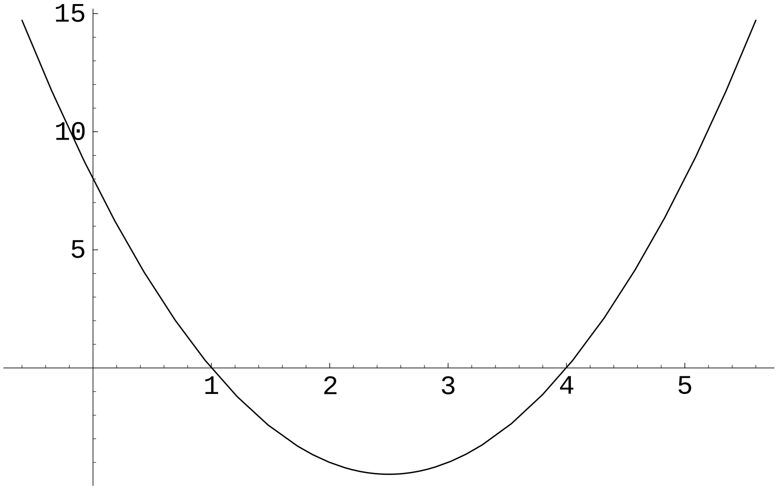
Problem 1 *(14 points)*
Graph the parabola $y = 2x^2 - 10x + 8$, clearly marking x- and y-intercepts and the vertex in your picture.
Solution: One approach is to factor it: $y = 2x^2 - 10x + 8 = 2(x^2 - 5x + 4) = 2(x-4)(x-1)$. We see that this is an upward-opening parabola with x-intercepts 1 and 4 (because those are the roots, as seen in the factored form). The vertex lies on the vertical line halfway between the roots: that is, the line $x = \frac{1+4}{2} = 2.5$. The y-coordinate of the vertex is

$$\begin{aligned} y &= 2(2.5-4)(2.5-1) \\ &= 2(-1.5)(1.5) \\ &= -3(1.5) \\ &= -4.5 \end{aligned}$$

So, the vertex is at $(2.5, -4.5)$. The y-intercept is 8 (most easily seen in the original equation). The graph:



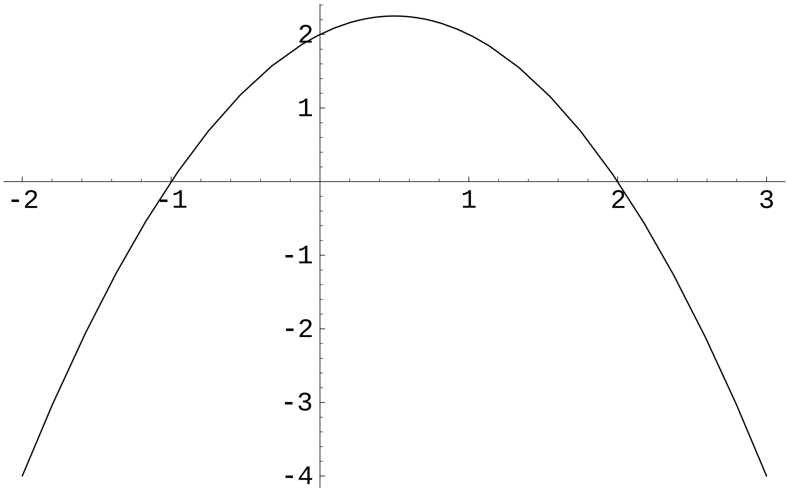
There's (of course) another approach, which we'll demonstrate on the alternate version of this problem: Graph the parabola $y = -x^2 + x + 2$, clearly marking x- and y-intercepts and the vertex in your picture.
Solution: we complete the square.

$$\begin{aligned} y &= -x^2 + x + 2 \\ &= -(x^2 - x \quad) + 2 \\ &= -(x^2 - x + \frac{1}{4} - \frac{1}{4}) + 2 \\ &= -(x^2 - x + \frac{1}{4}) - (-\frac{1}{4}) + 2 \\ &= -(x - \frac{1}{2})^2 + 2\frac{1}{4} \end{aligned}$$

We have a downward-opening parabola (because of the negative leading coefficient); the vertex is at $(\frac{1}{2}, 2\frac{1}{4})$. Since the vertex is above the x-axis and the parabola opens downward, we expect to find x-intercepts; we may locate them precisely by using the Quadratic Formula on the original equation, $y = -x^2 + x + 2$. The x-intercepts are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(-1)(2)}}{2(-1)} = \frac{-1 \pm \sqrt{1 + 8}}{-2} = \frac{-1 \pm 3}{-2} = 2, -1.$$

The graph:



Problem 2 *(14 points)*

What is the largest possible area for a right triangle in which the sum of the lengths of the two shorter sides is 80 cm?

Let's call the two shorter sides x and y. Since they are the legs of a right triangle, we can think of x as the base and y as the height. We are given that the sum of these two lengths is 80 cm, which is to say that $x + y = 80$. We may solve this equation for y, obtaining $y = 80 - x$. The area of the triangle is $\frac{1}{2}xy$ (one half base times height), which we may write in terms of x: $A = \frac{1}{2}xy = \frac{1}{2}x(80 - x) = -\frac{1}{2}x^2 + 40x$. Notice that if we were to graph this relationship between A and x, we would see a downward-opening parabola. The highest point on this parabola, which corresponds to the highest possible value for A (and therefore the solution to our problem) is its vertex. Either of the methods used on the previous problem would work just as well here; alternatively, we could use the shortcut formula that says the vertex has x-coordinate $\frac{-b}{2a} = \frac{-40}{2(-1/2)} = \frac{-40}{-1} = 40$. We now substitute 40 for x in $A = \frac{1}{2}x(80 - x)$ to obtain $A = \frac{1}{2}(40)(40) = 800$. The maximum possible area is 800 cm^2.

The other version of this problem used the total length "60 in." in place of "80 cm". The maximum area in that case (found via the same method) is 450 in^2.

Problem 3 *(16 points)*
Find all values of x such that $\log_2 x + \log_2(x+1) \leq 1$. Remember to think about the domain!
Solution: Let's start with the hint, and find the domain. In order for $\log_2 x$ to be defined, we must have $x > 0$. In order for $\log_2(x+1)$ to be defined, we must have $x+1 > 0$, or $x > -1$. But this is already true if $x > 0$, so we just use the constraint $\boxed{x > 0}$.
Now, let's use log rules to manipulate the inequality.

$$\begin{aligned} \log_2 x + \log_2(x+1) &\leq 1 \\ \log_2[x(x+1)] &\leq 1 \\ x(x+1) &\leq 2^1 \\ x^2 + x &\leq 2 \\ x^2 + x - 2 &\leq 0 \\ (x-1)(x+2) &\leq 0 \end{aligned}$$

To solve this polynomial inequality, we could notice that the key numbers are 1 and -2, and then test points on the left, on the right, and between these numbers. Or, we could notice that the parabola $y = (x-1)(x+2)$ sags below the x-axis between its roots (which are 1 and -2). So, the part of the x-axis which satisfies the condition $(x-1)(x+2) \leq 0$ is the interval $[-2, 1]$. The part of this interval that satifies our domain constraint (which, we recall, was $x > 0$) is the smaller interval $(0, 1]$. That is our final solution.
The other version of this problem uses the log base 6 instead of the log base 2. In that problem, the domain constraint is still $x > 0$, while the interval satisfying the polynomial inequality turns out to be $[-3, 2]$. The final solution for that version is $(0, 2]$.

Problem 4 *(12 points)*
Solve the equation

$$9^{x+2} = 3^{8x+7}.$$

The easiest approach is to realize that $9 = 3^2$. The equation becomes:

$$\begin{aligned}
9^{x+2} &= 3^{8x+7} \\
\left(3^2\right)^{x+2} &= 3^{8x+7} \\
3^{2(x+2)} &= 3^{8x+7} \\
3^{2x+4} &= 3^{8x+7} \\
2x+4 &= 8x+7 \\
4-7 &= 8x-2x \\
-3 &= 6x \\
-\frac{3}{6} &= x \\
x &= -\frac{1}{2}
\end{aligned}$$

The alternate version of this problem asks us to solve $9^{2x+1} = 3^{2x+5}$ instead. The method of solution is essentially the same, but we arrive at the answer $x = -\frac{3}{2}$.

Problem 5 *(10 points)*
Write the following logarithm as a sum or difference of simpler logarithms, such that no products, quotients, radicals, or powers occur inside a logarithm.
(Both versions of the problem are solved below.)

$$\begin{aligned}
\log \frac{x^2\sqrt{4-x}}{(2x-1)^{2/3}} &= \log(x^2\sqrt{4-x}) - \log((2x-1)^{2/3}) \\
&= \log(x^2) + \log(\sqrt{4-x}) - \frac{2}{3}\log(2x-1) \\
&= 2\log x + \frac{1}{2}\log(4-x) - \frac{2}{3}\log(2x-1)
\end{aligned}$$

$$\begin{aligned}
\log \frac{(x+2)^2\sqrt{x}}{(3x-2)^{3/2}} &= \log((x+2)^2\sqrt{x}) - \log((3x-2)^{3/2}) \\
&= \log(x+2)^2 + \log\sqrt{x} - \frac{3}{2}\log(3x-2) \\
&= 2\log(x+2) + \frac{1}{2}\log x - \frac{3}{2}\log(3x-2)
\end{aligned}$$

Problem 6 *(18 points)*
Graph the following rational function. Include all intercepts and asymptotes, as well as the precise coordinates of any point where the function crosses an asymptote. You may find it helpful to use other tools, too.

$$y = \frac{(x+2)(x-4)}{(x-2)(x+1)}$$

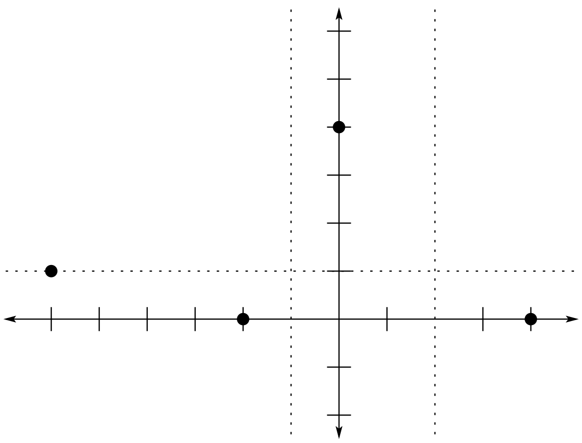
The fastest things to find are x-intercepts (where the numerator is zero) and vertical asymptotes (where the denominator is zero). The x-intercepts are -2 and 4, while the vertical asymptotes are the lines $x = 2$ and $x = -1$.
We find the y-intercept by setting $x = 0$; then $y = \frac{(0+2)(0-4)}{(0-2)(0+1)} = \frac{-8}{-2} = 4$.
Since the numerator and denominator are polynomials of the same degree, we expect a horizontal asymptote. If you know a shortcut, you may use it; however, to be thorough, I'll show the unabridged version. For large values of x,

$$\frac{(x+2)(x-4)}{(x-2)(x+1)} = \frac{x^2-2x-8}{x^2-x-2} = \frac{x^2(1-\frac{2}{x}-\frac{8}{x^2})}{x^2(1-\frac{1}{x}-\frac{2}{x^2})} = \frac{1-\frac{2}{x}-\frac{8}{x^2}}{1-\frac{1}{x}-\frac{2}{x^2}} \approx \frac{1-0-0}{1-0-0} = 1.$$

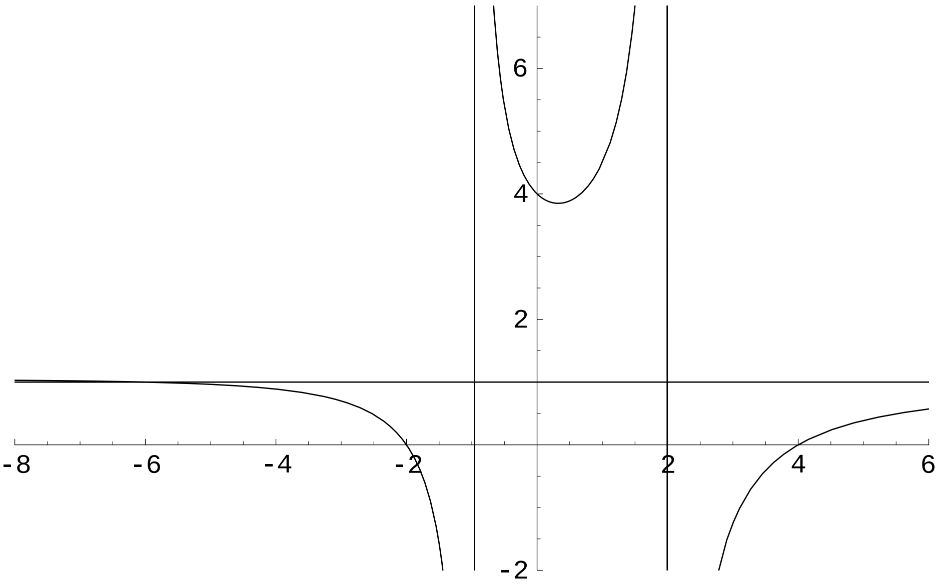
So, the horizontal asymptote is $y = 1$.
Where does the function cross its horizontal asymptote? We set the function and its asymptote equal to each other and solve for x:

$$\begin{aligned}
\frac{(x+2)(x-4)}{(x-2)(x+1)} &= 1 \\
(x+2)(x-4) &= (x-2)(x+1) \\
x^2-2x-8 &= x^2-x-2 \\
-2x-8 &= -x-2 \\
-8+2 &= -x+2x \\
-6 &= x
\end{aligned}$$

Since the point of intersection has to lie on the horizontal asymptote, we know the y-coordinate is 1. So, (-6,1) is the point of intersection. Let's see what we have so far:

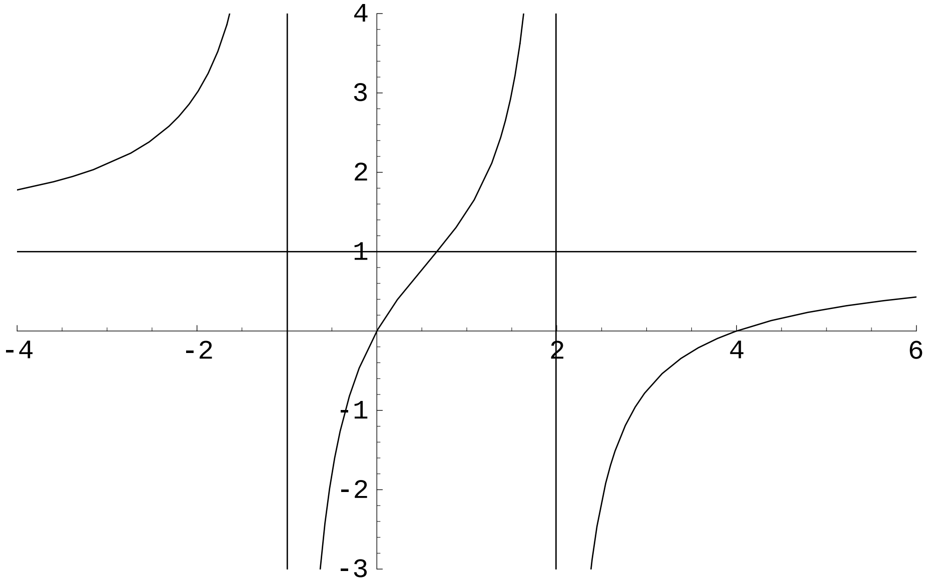


Our sketch includes the coordinate axes and three asymptotes; our curve can cross one of these lines *only* at the indicated points. This severely limits the directions the graph could go. For more detail, we could use the method of approximation for values of x near -2, -1, 2, and 4 if we wanted. The final graph should resemble...



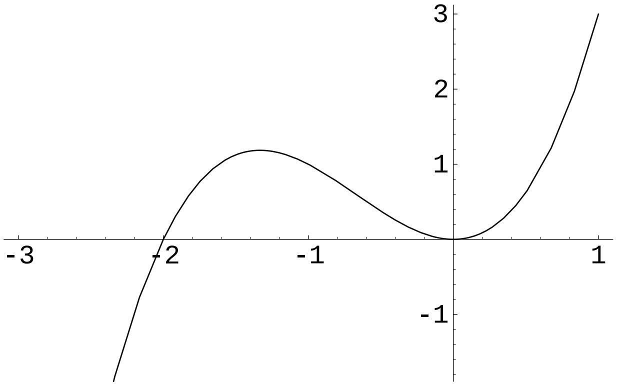
Although it's not shown on this graph, the left end of the curve peaks around (-12.32, 1.04) before slowly falling towards the asymptote again.

The alternate problem was to graph $y = \frac{x(x-4)}{(x-2)(x+1)}$:

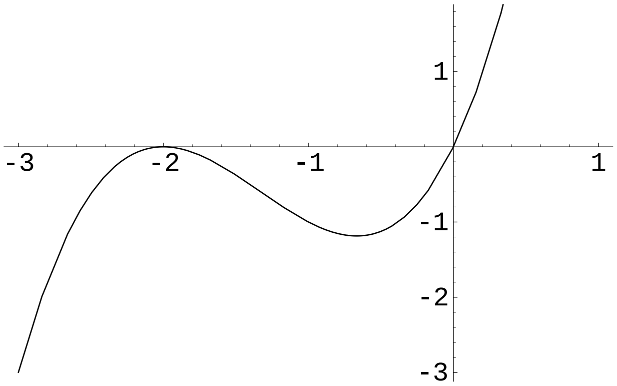


Problem 7 *(16 points)*

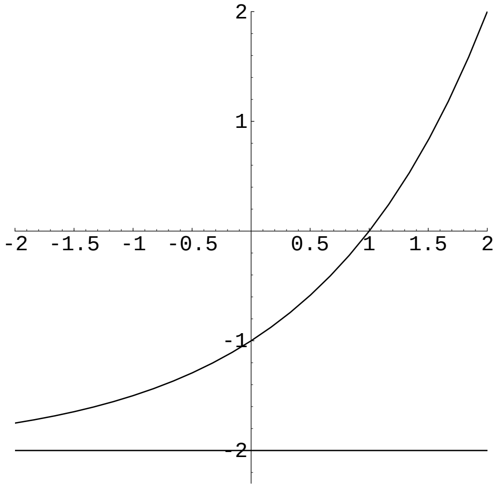
Match each function to its graph. Actually, I'll do the reverse.



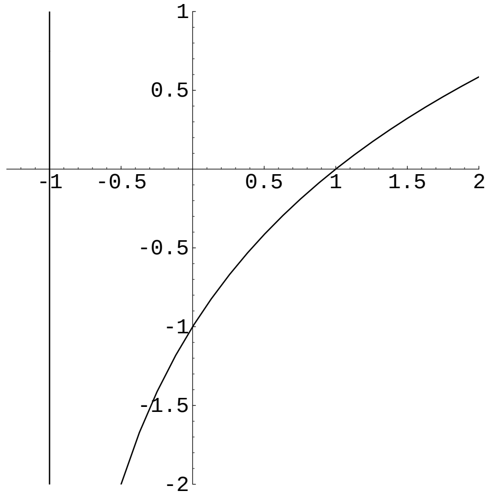
This function crosses the x-axis at $x = -2$ and bounces (for lack of a better word) like a parabola at $x = 0$; that is, -2 is a single root and 0 is a double root. The function is $y = x^2(x + 2)$.



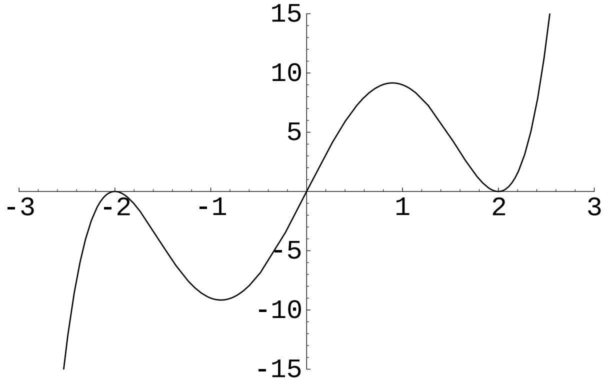
This is like the previous one, but the bounce (and therefore the double root) is at $x = -2$. The function is $y = x(x + 2)^2$.



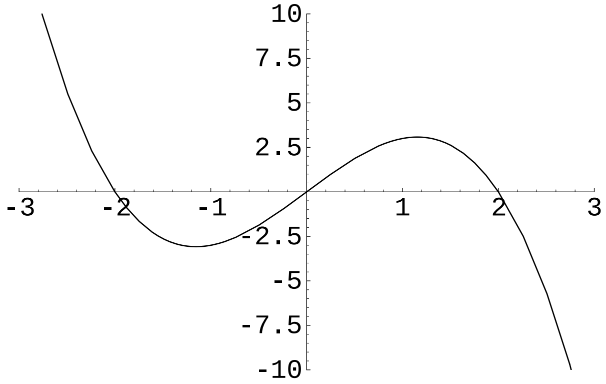
This graph has a horizontal asymptote, so it isn't a polynomial or a logarithm; that leaves the exponentials. Since the horizontal asymptote is $y = -2$, the function is $y = 2^x - 2$. (For confirmation, note that this exponential function grows without bound as x grows in the positive direction.)



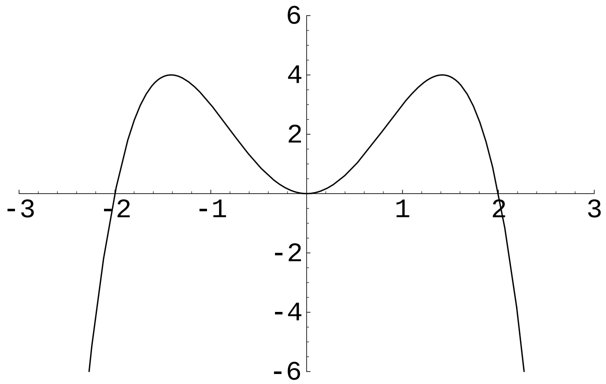
This is the only graph with a vertical asymptote, so it must be the log. More specifically, $y = \log_2(x+1) - 1$.



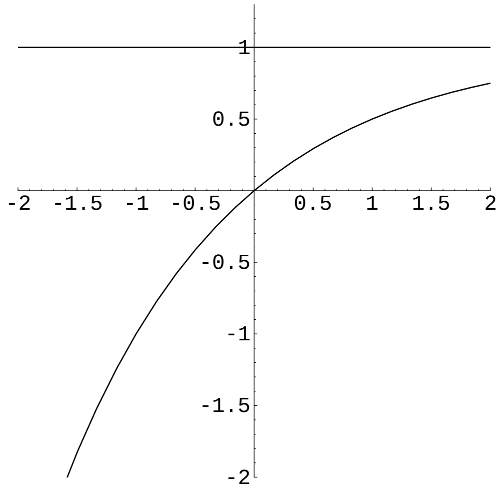
This function appears to be a quintic (fifth degree) polynomial or higher, because it has four turning points. (It looks almost like a slanted sine wave, but there were no such functions on the test.) Especially helpful are the double roots at $x = \pm 2$ and the single root at $x = 0$. This is the function $y = x(x+2)^2(x-2)^2$.



This function has the same roots as the last one, but all crossings are “transverse” (unlike how the previous function behaves near $x = \pm 2$). This indicates that all the roots are singles; that is, the function is $y = x(x+2)(-x+2)$.



This graph is a quartic (fourth degree) polynomial. Notice the double root at $x = 0$ and the single roots at $x = \pm 2$. This is $y = x^2(x + 2)(-x + 2)$. It may be helpful to multiply out this polynomial, and notice that the curve opens downward.



This is the only function left, $y = 1 - \left(\frac{1}{2}\right)^x$. It's a decaying exponential, which is to say that as x gets large (and positive), this function gets very tame—it approaches its horizontal asymptote ($y = 1$).