# SOLUTIONS to Math 12 Midterm 2, Winter 2004.

**Problem 1** *(10 points)*
Graph the parabola $y = -3x^2 + 6x + 4$, clearly marking $x$- and $y$-intercepts and the vertex in your picture.

When graphing a parabola, it helps to have the equation in a few different forms so that we can get information from each form. First, let's complete the square:

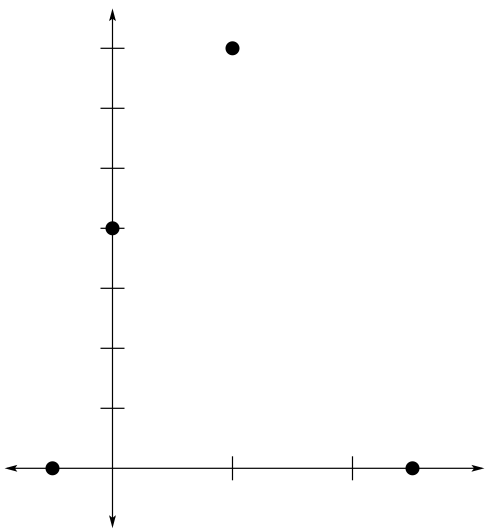
$$\begin{aligned} y &= -3x^2 + 6x + 4 \\ &= -3(x^2 - 2x) + 4 \\ &= -3(x^2 - 2x + 1 - 1) + 4 \\ &= -3(x^2 - 2x + 1) - 3(-1) + 4 \\ &= -3(x-1)^2 + 3 + 4 \\ y &= -3(x-1)^2 + 7 \end{aligned}$$

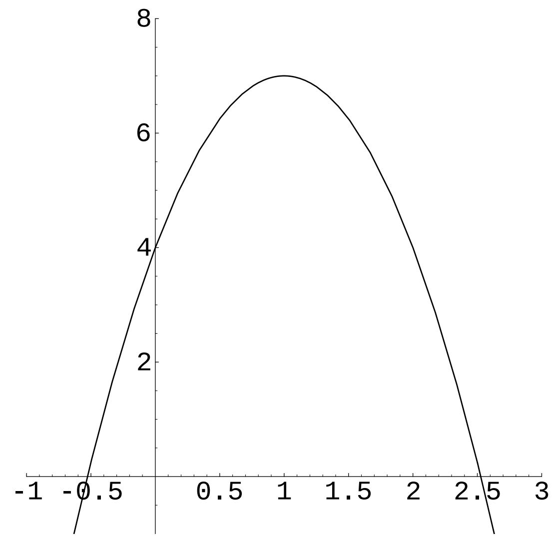
So, we see that $\boxed{\text{this parabola has vertex (1,7) and opens downward}}$. To find the $y$-intercept, we substitute 0 for $x$ in the equation (perhaps the original is easiest), and we get $y = 4$. Thus, $\boxed{\text{the } y\text{-intercept is at } y = 4}$. To get $x$-intercepts, we should factor it. This one turns out to be difficult to factor, so rather than mess around with that for too long, let's substitute 0 for $y$ in the last equation above. Now we solve that equation:

$$\begin{aligned} 0 &= -3(x-1)^2 + 7 \\ 3(x-1)^2 &= 7 \\ (x-1)^2 &= \frac{7}{3} \\ x - 1 &= \pm\sqrt{\frac{7}{3}} \\ x &= 1 \pm \sqrt{\frac{7}{3}} \end{aligned}$$

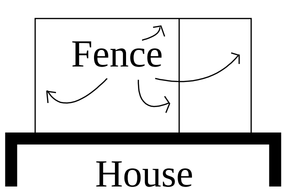
Now, 7/3 is a little more than 2, so $\sqrt{7/3}$ is a little more than $\sqrt{2} \approx 1.4$. Since we're graphing by hand, we aren't worried about perfect accuracy; it's enough to know that $\sqrt{7/3}$ is somewhere between 1 and 2. We'll use the approximation 1.5

just to help keep the numbers under control. Now, $1+\sqrt{7/3} \approx 1+1.5 = 2.5$, while $1-\sqrt{7/3} \approx 1-1.5 = -0.5$. So we mark those as $x$-intercepts, and label them $\boxed{1+\sqrt{7/3} \text{ and } 1-\sqrt{7/3}}$, respectively. (Incidentally, you could find those $x$-intercepts by way of the Quadratic Formula instead of how I did it, if you like.) Here we have two pictures: first, the coordinate axes with the vertex and intercepts marked; second, the graph according to Mathematica.



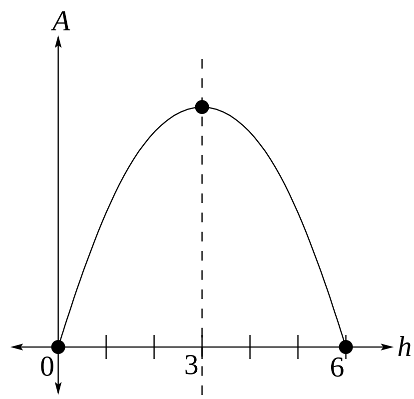



**Problem 2** *(15 points)*
I want to keep chickens and rabbits in a divided rectangular pen in my back yard, adjacent to my house (see the diagram below). I have 18 yards of fencing material. What is the largest area I can enclose? (Please show your strategy as well as your solution.)




First, we should label the diagram with helpful variables. One choice is to name the vertical pieces of fence $h$, and the horizontal piece $w$. There are three parts of length $h$ and one part of length $w$, so the total amount of fence used is $3h+w$. Of course, we used all available fence (to maximize area), so we can say $3h+w=18$. Next, we want to know how much area is enclosed. Well, the area of the pen is

$A = wh$. It would help if we had $A$ expressed in terms of a single variable. Which variable we use only makes a little difference; I'll do it both ways so you can see.
If we solve the equation $3h + w = 18$ for $w$, it becomes $w = 18 - 3h$. We now use this to substitute for $w$ in $A = wh$, and we get $A = (18 - 3h)h = -3h^2 + 18h = -3h(h - 6)$. If we were to graph $A$ versus $h$, we would get a downward-pointing parabola. Now, we could complete the square at this point to locate the vertex, or we could use the $\frac{-b}{2a}$ shortcut, but we're going to do something else. (I'm not saying it's better; those other things would work too.) Notice that $A = -3h(h - 6)$ has roots at $h = 0, 6$. So, the axis of symmetry must be $h = 3$ (which is the midpoint, or average, of 0 and 6). Like this:




The vertex is the only point of the parabola on the axis of symmetry, so we find the maximal value of $A$ by putting 3 in for $h$ in one of our equations relating $h$ to $A$. We find $A = -3(3)(3-6) = 27$. In other words, the maximal area is $\boxed{\text{27 square yards}}$, and it occurs when $h = 3$ yards. Just to be thorough: $w = 18 - 3(3) = 9$ yards, and sure enough, a 3 yard by 9 yard pen is 27 square yards.
I said I would do this "both" ways, so here's number 2. Instead of solving for $w$ in $3h + w = 18$, we'll solve for $h$. This has the unpleasant side effect that we'll get a fraction, but we can take it.

$$\begin{aligned} 3h + w &= 18 \\ 3h &= 18 - w \\ h &= 6 - \frac{w}{3} \end{aligned}$$

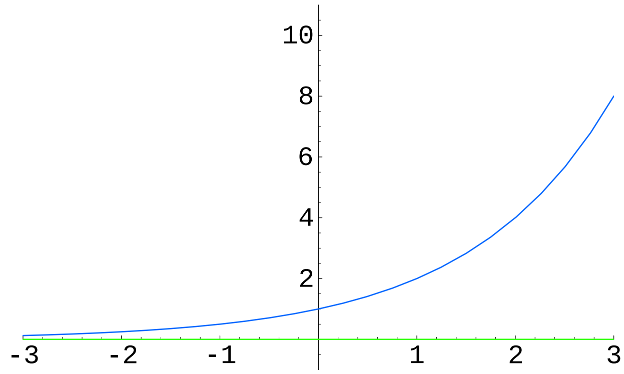
Now we drop this formula into $h$'s place in the equation $A = wh$, and we get: $A = w(6 - \frac{w}{3})$. We could certainly finish this the way we did the previous one, but I feel like completing the square this time. So, here we go.

$$
\begin{aligned}
A &= w(6 - \frac{w}{3}) \\
&= 6w - \frac{1}{3}w^2 \\
&= -\frac{1}{3}w^2 + 6w \\
&= -\frac{1}{3}(w^2 - 18w \quad ) \\
&= -\frac{1}{3}(w^2 - 18w + 81 - 81) \\
&= -\frac{1}{3}(w^2 - 18w + 81) - \frac{1}{3}(-81) \\
A &= -\frac{1}{3}(w - 9)^2 + 27
\end{aligned}
$$

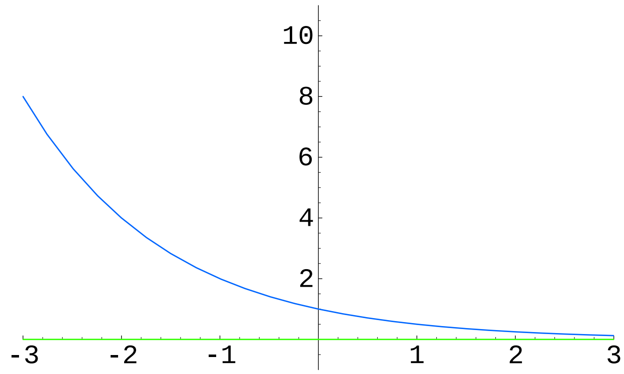
Some of you have your own ways of completing the square, and that's just fine—but we should all arrive at the same formula at the end. We see from the last equation above that a graph of $A$ versus $w$ would be a downward-pointing parabola with vertex (9,27); this means that the maximal area of 27 square yards will occur when $w = 9$ yards. We may also calculate $h = 6 - \frac{9}{3} = 6 - 3 = 3$ yards. These measurements all coincide with the measurements we got previously.

**Problem 3** *(10 points)*
Graph $f(x) = 2 + 2^{-x}$. Indicate asymptotes and intercepts. What is the range of $f$?
It is helpful in this case to consider a *sequence* of graphs: $y = 2^x$, $y = 2^{-x}$, $y = 2 + 2^{-x}$. Hopefully, the first of these is familiar; but if not, we can generate it by plotting points without much difficult calculation. The second curve is obtained by reflecting the first across the $y$-axis. (Why not the $x$-axis? See p.131 and the examples that follow. Actually, don't just look at them, *reinact* them, until they make sense to you.) Finally the third curve is obtained from the second by a two-unit upward slide. We can now see that our graph does not actually touch the line $y = 2$, but it gets very close; also, it stretches as far above that line as you like. In the graphs below, the curve itself is drawn in blue, while the horizontal asymptote is drawn in green:

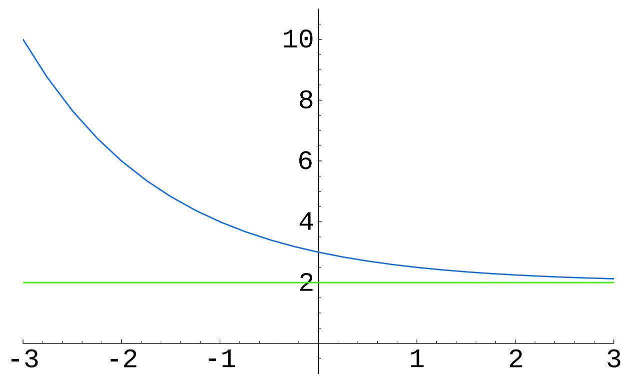
$y = 2^x$




$y = 2^{-x}$




$y = 2 + 2^{-x}$




The range (of the last curve shown above) is $(2, \infty)$.

**Problem 4** *(15 points)*
What is the domain of $f(x) = \log_4(2x - 3)$?
We can take logs of only positive numbers. So, the stuff inside the log, namely $2x - 3$, must be positive; thus,

$$\begin{aligned} 2x - 3 &> 0 \\ 2x &> 3 \\ x &> \frac{3}{2} \end{aligned}$$

Our domain is all $x > 3/2$, or the interval $(\frac{3}{2}, \infty)$.
Find all values of $x$ such that $\log_4(2x - 3) \le 1$.
We would like to express the inequality without using logarithms. I'll show you three—count them—*three* ways to do this.
Method 1: "Exponential Form". Recall that the logarithmic equation $\log_4(2x-3) = 1$ can be put into *exponential form* as $2x - 3 = 4^1$. We can do the same to the inequality; if $\log_4(2x-3) \le 1$, then $2x - 3 \le 4^1$. When doing it this way, you have to be careful that the base of the log is the only thing to move across the $\le$ sign. Otherwise you might get the inequality backwards.
Method 2: "Raise four to both sides". Here we use the property that if $a \le b$, then $4^a \le 4^b$ (see p.315, property 1a). Since we are given that $\log_4(2x - 3) \le 1$, we may conclude that

$$\begin{aligned} 4^{\log_4(2x-3)} &\le 4^1 \\ 2x - 3 &\le 4 \end{aligned}$$

Note that $4^x$ and $\log_4 x$ are each other's inverse functions, so you could say the $4^{\cdot}$ and the $\log_4 \cdot$ cancel each other out, leaving us with just the $2x - 3$ on the left-hand side.
Method 3: "Write 1 in a more convenient form". First we use the fact that $\log_4 4 = 1$, and then we use property 2b from p.315:

$$\begin{aligned} \log_4(2x - 3) &\le 1 \\ \log_4(2x - 3) &\le \log_4 4 \\ 2x - 3 &\le 4 \end{aligned}$$

Note that both logs have base 4.
However we do it, we get to the inequality $2x - 3 \le 4$, so $2x \le 7$; thus, $\boxed{x \le 7/2}$. This isn't the final answer, however. Note that $1 \le 7/2$, so this would suggest that 1 is a solution to the inequality we're trying to solve. The problem is that, plugging 1 in for $x$, we get $\log_4(-1)$, which is undefined. We must use the inequality $x \le 7/2$ together with the domain constraint $x > 3/2$, giving us the actually correct final solution, $\boxed{\frac{3}{2} < x \le \frac{7}{2}}$. We could also write this as the interval $(\frac{3}{2}, \frac{7}{2}]$.

**Problem 5** *(15 points)*
Find all roots and asymptotes of the following rational function. Find the equation of its horizontal asymptote, and find all points at which your function crosses this

horizontal asymptote. Sketch a graph of this function which takes into account all of this information, as well as any other analysis you find helpful.

$$y = \frac{(x-1)^2}{(x+2)(x+4)}.$$

Roots, a.k.a. $x$-intercepts, happen when the numerator is zero—that is, when $x = 1$. In fact, the exponent of 2 upstairs indicates that $x = 1$ is a *double root.* We'll see the significance of this later.

Vertical asymptotes happen when the denominator is zero; i.e., when $x = -2, -4$.

Oh, I didn't ask you to find the $y$-intercept in the problem, but you know you want to. We set $x = 0$, and we find

$$y = \frac{(0-1)^2}{(0+2)(0+4)} = \frac{1}{8}.$$

So, the $y$-intercept happens at $\boxed{y = \frac{1}{8}}$.

Now we get to the horizontal asymptote. We rewrite the function as

$$y = \frac{x^2 - 2x + 1}{x^2 + 6x + 8} = \frac{x^2(1 - \frac{2}{x} + \frac{1}{x^2})}{x^2(1 + \frac{6}{x} + \frac{8}{x^2})} = \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{6}{x} + \frac{8}{x^2}}.$$

For large values of $x$, all the little fractions involving $x$ become very very small. We ignore them; the value of $y$ gets very close to $\frac{1}{1} = 1$. So, the horizontal asymptote is $\boxed{y = 1}$. If that was too strange for you, try long division instead. Since I don't know how to typeset long division, I'll just tell you that you should get a quotient of 1, remainder $-8x - 7$. Another way to put that:

$$\begin{aligned} y &= \frac{(x-1)^2}{(x+2)(x+4)} \\ &= \frac{x^2 - 2x + 1}{x^2 + 6x + 8} \\ &= \frac{x^2 + 6x + 8 - 8x - 7}{x^2 + 6x + 8} \\ &= \frac{x^2 + 6x + 8}{x^2 + 6x + 8} - \frac{8x + 7}{x^2 + 6x + 8} \\ &= 1 - \frac{8x + 7}{x^2 + 6x + 8} \end{aligned}$$

(This is a top secret ninja technique. If you want to master it, try working your way up or down from the third line in that sequence of equations.) Point is, we find the horizontal asymptote by dropping the remainder. If you think long division and top secret ninja techniques are too fancy, you may be right—this time. But they are **great** for finding slant asymptotes and even *asymptotes that aren't lines.* Don't freak, I won't ask you about bendy asymptotes.

Let's find out where our rational function crosses the horizontal asymptote. That's right, A Function Can Cross An Asymptote. In general, to find where two functions meet, set them equal to each other. Setting the rational function equal to 1, we get:

$$\begin{aligned}
\frac{x^2-2x+1}{x^2+6x+8} &= 1 \\
x^2-2x+1 &= x^2+6x+8 \\
-2x+1 &= 6x+8 \\
1 &= 8x+8 \\
-7 &= 8x \\
-7/8 &= x
\end{aligned}$$

Thus, $x=-\frac{7}{8}$ is the only point where our function crosses its own asymptote.

If we draw the information we've collected, we may be able to figure out how the graph looks. But first, I'd like to get one more thing: a sign chart. The only places where our rational function might change sign are across roots and vertical asymptotes; that is, the values $x=-4,-2,1$ break the number line into 4 intervals (two finite, two infinite). On each of these intervals, the function is always positive or always negative. Our job is to decide which. We can use test points $-10,-3,0$, and 5 (one of these numbers lies in each of the intervals). When we substitute these for $x$, what matters is NOT the actual number we get for $y$, but whether it is positive or negative. Here, see what I mean:

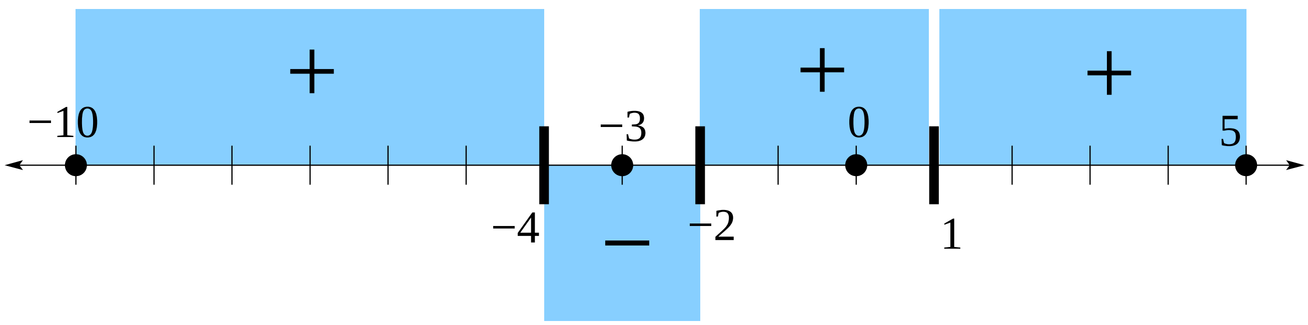
At $x=-10$, $y=\frac{(-11)^2}{(-8)(-6)}=\frac{(-)^2}{(-)(-)}=\frac{+}{+}=+$.

At $x=-3$, $y=\frac{(-4)^2}{(-1)(1)}=\frac{(-)^2}{(-)(+)}=\frac{+}{-}=-$.

At $x=0$, $y=\frac{(-1)^2}{(2)(4)}=\frac{(-)^2}{(+)(+)}=\frac{+}{+}=+$.

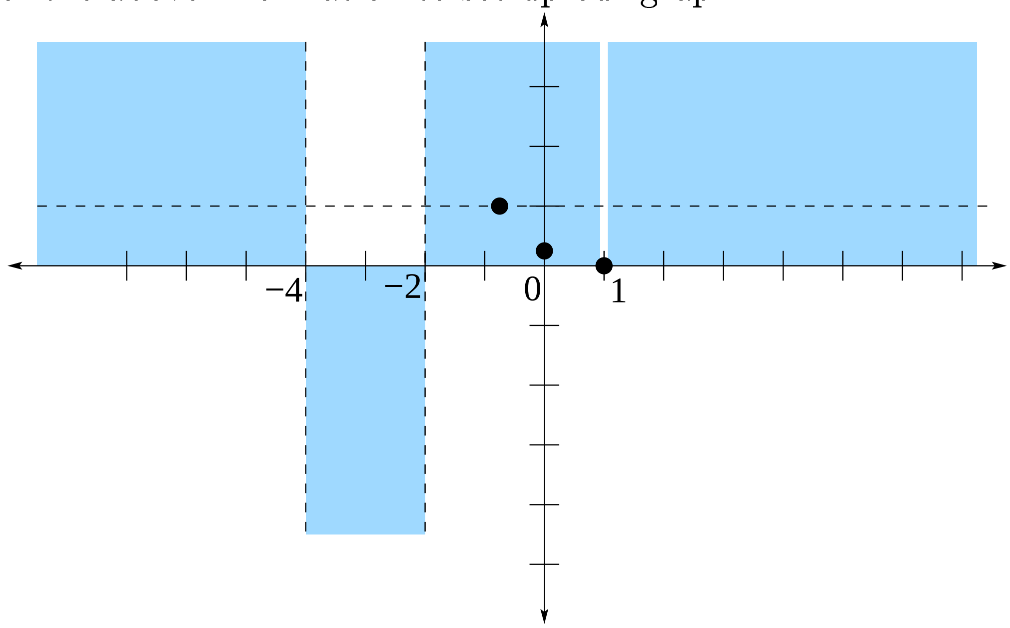
At $x=5$, $y=\frac{(4)^2}{(7)(9)}=\frac{(+)^2}{(+)(+)}=\frac{+}{+}=+$.

The image below is our sign chart, where the blue part indicates where the function has to lie.

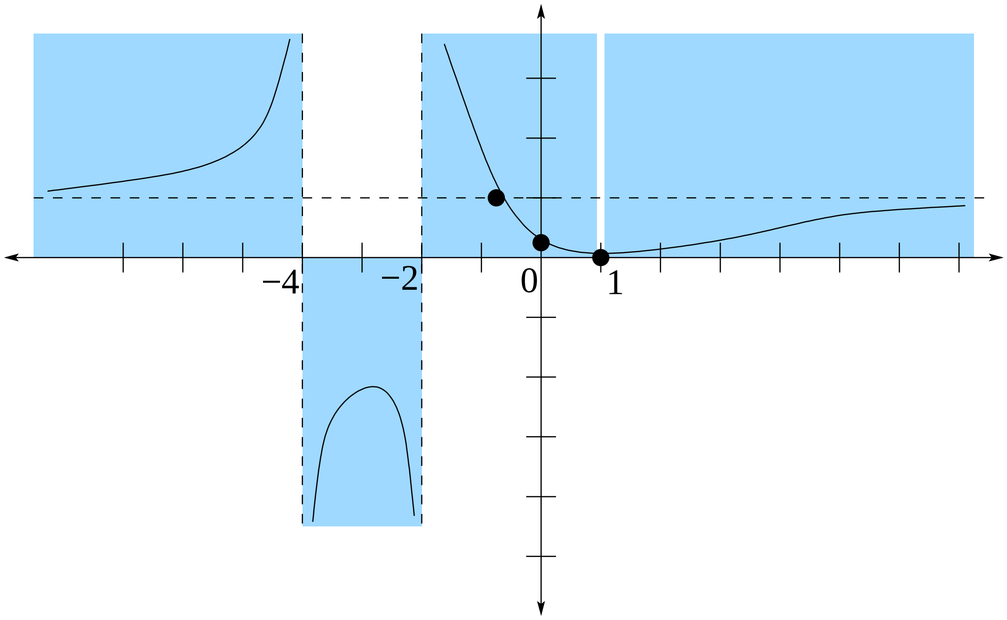



We use all of the above information to set up our graph.

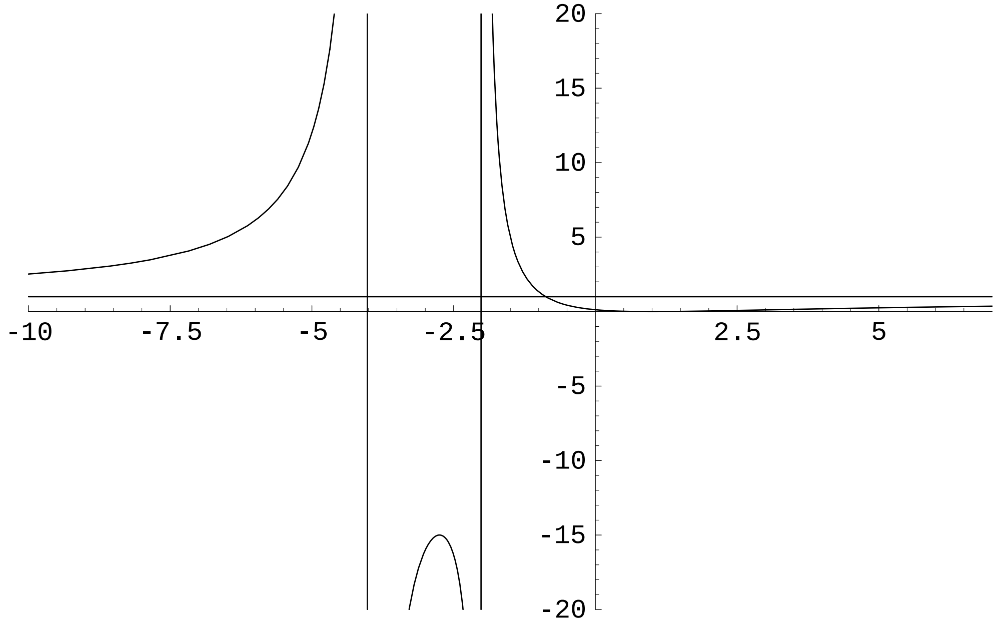
The setup:



The sketch:




The graph according to Mathematica:




Note that some parts of our sketch are out of scale. For instance, all the interesting behavior is nicely visible in our sketch—but in reality, if we want to see the bump between the asymptotes, we have to use such a large window that it is difficult to see what's happening between the horizontal asymptote and the $x$-axis. Also, we DO NOT KNOW exactly where the top of that bump is.

**Problem 6** *(15 points)*
Solve the equation $5^{2x} + 5^x - 6 = 0$.
In order to have a chance at this, we really need to know that $5^{2x} = (5^x)^2$. Now we let $t = 5^x$, and substitute:

$$\begin{aligned} 5^{2x} + 5^x - 6 &= 0 \\ (5^x)^2 + 5^x - 6 &= 0 \\ t^2 + t - 6 &= 0 \\ (t-2)(t+3) &= 0 \end{aligned}$$

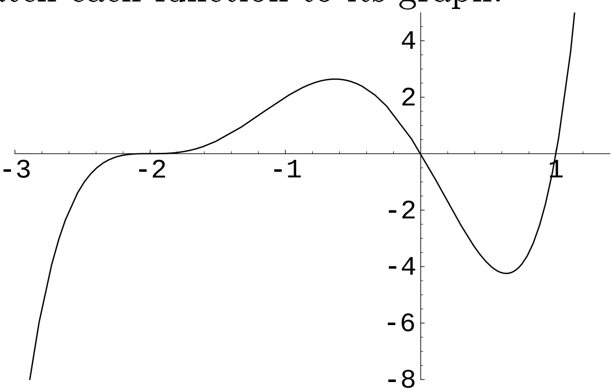
From this we conclude that $t = 2$ or $t = -3$. Is this the answer? Well, we were supposed to solve an equation involving $x$, not $t$. We won't be done until we have an answer for $x$. So we substitute back. If $t = 2$, then $5^x = 2$. We express this exponential equation in logarithmic form: $\boxed{x = \log_5 2}$. And for the other solution: if $t = -3$, then $5^x = -3$. Trouble is, $5^x$ cannot be a negative number, so the solution $t = -3$ does not correspond to any value of $x$. That is, the boxed solution is the *only* solution.

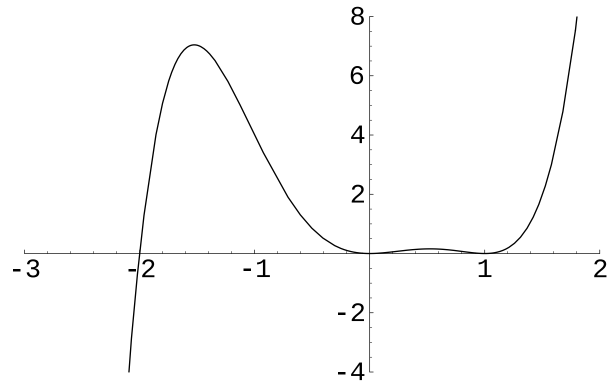
**Problem 7** *(10 points)*
Write as a single logarithm: $3\log x - 2\log(x+1) + \frac{1}{2}\log(x+2)$
Study tip: Try to be fluent in the rules of logarithms. Confirm each property in the box on p.301 with an example, and likewise check each "error to avoid" in the box on p.306 with an example of your choosing.
Now for the solution to the current problem:

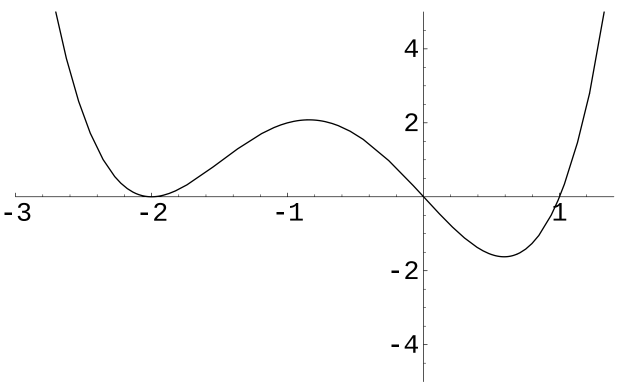
$$\begin{aligned} 3\log x - 2\log(x+1) + \frac{1}{2}\log(x+2) &= \log x^3 - \log(x+1)^2 + \log(x+2)^{1/2} \\ &= \log x^3 - \log(x+1)^2 + \log\sqrt{x+2} \\ &= \log\frac{x^3\sqrt{x+2}}{(x+1)^2} \end{aligned}$$

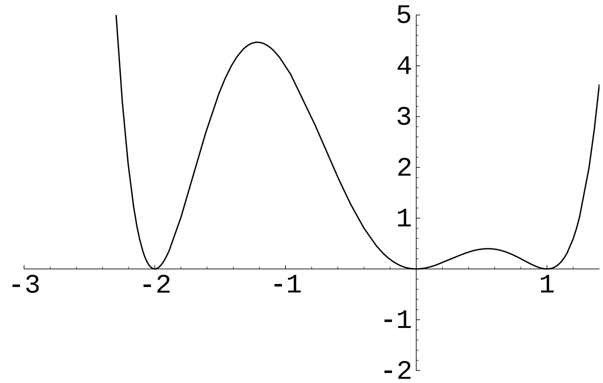
With practice, you will be able to do all of that in a single step.

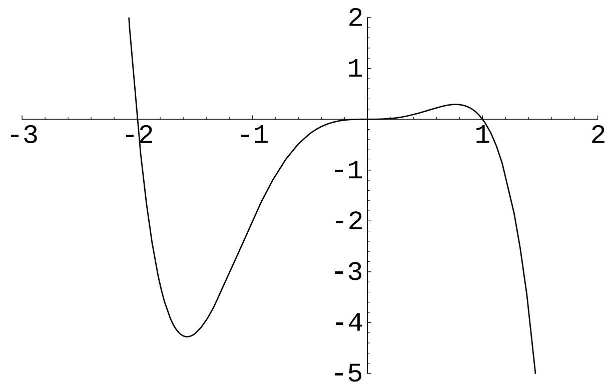
**Problem 8** *(10 points)*
Match each function to its graph.

A) 

B) 

C) 

D) 

E) 

_____ $y = (1 - x)(x + 2)x^3$

_____ $y = (x - 1)^2(x + 2)^2x^2$

_____ $y = (x - 1)(x + 2)^2x$

_____ $y = (x - 1)^2(x + 2)x^2$

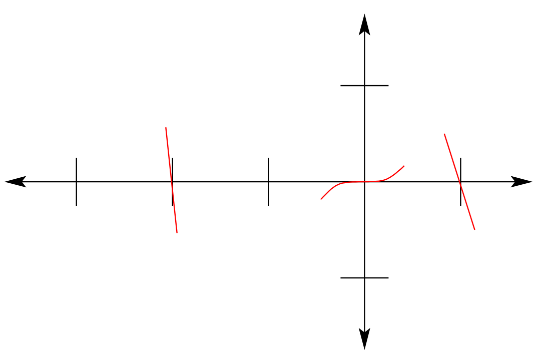
_____ $y = (x - 1)(x + 2)^3x$

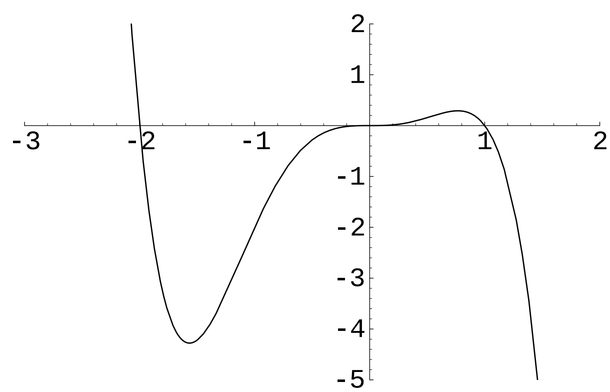
And now for my favorite saying: there are a few ways to think about this.

Method 1: The Approximation Technique. Consider the first function on the list, $y = (1-x)(x+2)x^3$. The "sensitive points" are $x = 1, -2, 0$, because these are the values of $x$ which make the whole function equal 0. (The "sensitive points" on a rational function will also include points where the function is undefined, i.e. where the denominator is zero.)

- Near $x = 1$, we approximate $y$ by setting $x = 1$ in all quantities *except* the sensitive one: $y \approx (1-x)(1+2)1^3 = 3(1-x) = -3x+3$. This is a descending line (slope -3) with $x$-intercept 1. We don't actually care where its $y$-intercept is, because that's too far from where we are focussing: $x \approx 1$.
- Near $x = -2$, we approximate $y$ by setting $x = -2$ in all quantities except the sensitive one: $y = (1-(-2))(x+2)(-2)^3 = 3(x+2)(-8) = -24(x+2)$. This is a *steeply* descending line (slope -24) with $x$-intercept -2.
- Near $x = 0$, we approximate $y$ by setting $x = 0$ in all quantities except the sensitive one: $y = (1-0)(0+2)x^3 = 2x^3$. This is a cubic polynomial with $x$-intercept 0 (it's a double–tall version of $y = x^3$).

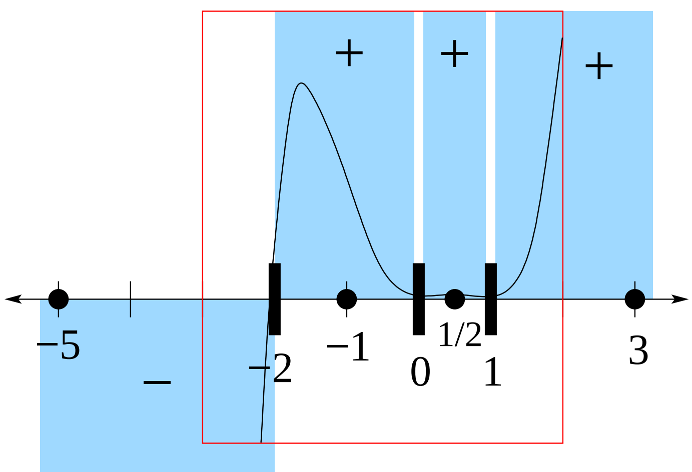
There is only one graph which resembles all three of these approximations near the specified points. Here we see, side by side, a sketch of the three described approximations and the correct function:

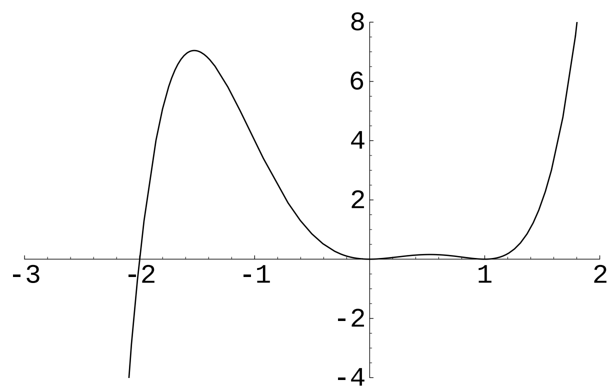





Secret ninja shortcut: since the factor $(1-x)$ has the (implicit) exponent 1, $x = 1$ is a single root. Since the factor $(x+2)$ has an implicit exponent of 1, $x = -2$ is also a single root. Since the factor $x$ has the exponent 3, we call $x = 0$ a triple root. This indicates that the function will resemble a line, a line, and a cubic (respectively) near these roots. Try to apply this technique to the other functions/graphs.

Method 2: The Sign Chart. Consider the function $y = (x-1)^2(x+2)x^2$. We note that it has roots $x = 1, -2, 0$. These numbers break the number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$. In each interval, we pick a sampling number. Let's choose the sampling numbers $-5, -1, \frac{1}{2}$, and 3.
At $x = -5$, $y = (-5-1)^2(-5+2)(-5)^2 = (-)^2(-)(-)^2 = (+)(-)(+) = -$.
At $x = -1$, $y = (-1-1)^2(-1+2)(-1)^2 = (-)^2(+)(-)^2 = (+)(+)(+) = +$.
At $x = \frac{1}{2}$, $y = (\frac{1}{2}-1)^2(\frac{1}{2}+2)(\frac{1}{2})^2 = (-)^2(+)(+)^2 = (+)(+)(+) = +$.
At $x = 3$, $y = (3-1)^2(3+2)(3)^2 = (+)^2(+)(+)^2 = (+)(+)(+) = +$.
Only one graph is below the $x$-axis on the left-most interval and above it on the remaining intervals. Here we have the sign chart and the matching function. The red box indicates the viewing rectangle used in the Mathematica graph.




I was going to tell you about Method 3: The Kitchen Sink. But then I had to go do other stuff.