Solutions to Math 12 Midterm 1 Review Packet

The correct way to use a solution set like this is NOT to simply read the problem and solution, and decide you understand what I've written. You should try the problems on your own a few times. If you reach an answer, try to verify independently that it is correct. If you try a problem from several different angles and still cannot get it to work, or if you just want to see a different correct answer than your own, that's a good time to check this solution set.

Problem 1 If possible, find:

- an integer that is not a natural number

  There are many valid answers, including 0, -1, -2, and so on.

- a real number than is not rational

  There are the classic examples: $\sqrt{2}$, e, and π. Also, if the natural number n is not a perfect square, then $\sqrt{n}$ is irrational (which is to say real and not rational). It may seem that these examples are very particular, perhaps even rare, but the bizarre truth of the matter is that there are many more irrational numbers than rational ones.

- a natural number that is not rational

  Sorry, no chance. Every natural number is rational (meaning, it can be written as the ratio of two integers); for example, $4 = \frac{4}{1}$.

- a rational number that is not an integer

  Some typical examples are $\frac{3}{2}$ or 4.2 or 2.44444444....

- an integer that is also rational

  Any integer you write down is also rational. So, -2 or 1 or 0 or 351994.

Problem 2 What is larger, $2 + \sqrt{3}$ or $3 + \sqrt{2}$?

The larger number is $3+\sqrt{2}$. To see why, note that both $\sqrt{2}$ and $\sqrt{3}$ are between 1 and 2. So, $2 + \sqrt{3}$ is between 3 and 4, while $3 + \sqrt{2}$ is between 4 and 5.

A more involved approach might look like this:

$$\begin{aligned}
2+\sqrt{3} &\ ?\ 3+\sqrt{2} \\
\sqrt{3} &\ ?\ 1+\sqrt{2} \quad \leftarrow \text{Both numbers are positive, so squaring preserves order.} \\
3 &\ ?\ 1+2\sqrt{2}+2 \\
0 &< 2\sqrt{2}
\end{aligned}$$

Now we reverse our steps:

$$\begin{aligned}
0 &< 2\sqrt{2} \quad \leftarrow \text{Add 3 to each side} \\
3 &< 1+2\sqrt{2}+2 \quad \leftarrow \text{Factor the right side} \\
3 &< (1+\sqrt{2})^2 \quad \leftarrow \text{Square root both sides} \\
\sqrt{3} &< 1+\sqrt{2} \quad \leftarrow \text{Add 2 to both sides} \\
2+\sqrt{3} &< 3+\sqrt{2}
\end{aligned}$$

There are some technical points to which I've alluded in the middle of this argument, but we won't get into that right now.

Problem 3 Simplify $|x-4|+|x+1|$, given that $-1 < x < 4$.

Since $-1 < x < 4$, we know two things. First, $x < 4$, so $x-4 < 0$, so $x-4$ is negative; thus, $|x-4| = -(x-4)$. Second, $x > -1$, so $x+1 > 0$, so $x+1$ is positive; that is, $|x+1| = x+1$. So, now we can say:

$$|x-4|+|x+1| = -(x-4)+(x+1) = -x+4+x+1 = 5.$$

The answer is 5.

Problem 4 **(a)** The inequality $x^2+3 \leq 2$ has no solution. Why not? Carefully explain.

Since x^2 is at least 0 for any choice of x, the left hand side cannot be any smaller than 3. Therefore it cannot be less than or equal to 2.

(b) Write an inequality of your own that has no solution.

There are several possibilities. An easy one is $x^2 < 0$. A little trickier: $x^4+1 < x^3$.

(c) Can you write an inequality that has exactly one solution? Exactly two?

Exactly one solution: $x^2 \leq 0$. Exactly two solutions: $(x-1)^2(x+1)^2 \leq 0$. Exactly three solutions: $(x-1)^2(x-2)^2(x-3)^2 \leq 0$. Why do these work?

Problem 5 "The water level of Rollins Lake is always about 2130 feet above sea level, but is sometimes above or below this elevation by up to 3 feet." Rewrite this sentence as an inequality, using absolute value, with the variable x representing the height (in feet) of the surface of Rollin's Lake above sea level. Solve your inequality for x.

We might rephrase the sentence as "The water level at Rollins Lake is within 3 feet of 2130." Expressed as an inequality, that is $|x - 2130| \leq 3$. We solve it like this:

$$\begin{array}{rcccl} & & |x-2130| & \leq & 3 \\ -3 & \leq & x-2130 & \leq & 3 \qquad \leftarrow \text{Add 2130 to all three "sides"} \\ 2127 & \leq & x & \leq & 2133 \end{array}$$

In interval notation, that's $[2127, 2133]$.

Problem 6 Write an inequality that expresses the fact that x is at least 5 units away from 12. Solve this inequality for x and write the solution set in interval notation.

The inequality is $|x - 12| \geq 5$. ("The distance between x and 12 is at least 5.") We solve it like this:

Since $|x - 12| \geq 5$, we know that either $x - 12 \geq 5$ or $x - 12 \leq -5$. So, either $x \geq 17$ or $x \leq 7$. In interval notation, that's $(-\infty, 7] \cup [17, \infty)$.

Problem 7 Draw on the number line the solution set of the inequality $|x - 2| \geq 3$.

These are all the points at least three units away from 2. So we center at 2, go three units in each direction, and keep going. See Figure 1.

Problem 8 For each of the following expressions, find the domain and represent it in interval notation:

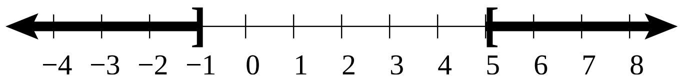


Figure 1: The solution set for $|x - 2| \geq 3$.

- $\sqrt{4-x}$

  Since $4 - x \geq 0$, we have the domain $x \leq 4$ or $(-\infty, 4]$.

- $\sqrt{4-|x|}$

  Since $4 - |x| \geq 0$, we have $|x| \leq 4$, so $-4 \leq x \leq 4$. That is, $[-4, 4]$ is our domain.

- $\sqrt{4-x^2}$

  We must have $4 - x^2 \geq 0$, so $x^2 \leq 4$, so $-2 \leq x \leq 2$. That is, our domain is $[-2, 2]$.

- $\frac{(x+1)(x-3)}{(x-2)(x+4)}$

  Here the only restriction is that we can't divide by zero, so x can be any real number except 2 or -4. Our domain, in interval notation, is $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$.

- $x^2 + 3x + 2$

  This is a polynomial; its domain is all reals, also known as $(-\infty, \infty)$.

- $\frac{1}{x^2+3x+2}$

  Here again we must avoid division by zero. So, we factor the denominator: $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)}$. Now x cannot be -1 or -2, and our domain is all other real numbers: $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

Problem 9 Factor the following expressions as far as you can:

- $x^2 - a^2 = (x - a)(x + a)$
- $x^2 + a^2$ is already factored.
- $x^3 + 1 = (x + 1)(x^2 - x + 1)$
- $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

- $2x^3 - 9x^2 - 5x = x(2x^2 - 9x - 5) = x(2x+1)(x-5)$

Problem 10 **(a)** Find all real solutions to the equation $\sqrt{x+2} = x - 4$.

We start by squaring both sides:

$$\begin{aligned}
\sqrt{x+2} &= x-4 \\
x+2 &= (x-4)^2 \\
x+2 &= x^2 - 8x + 16 \\
0 &= x^2 - 9x + 14 \\
x^2 - 9x + 14 &= 0 \\
(x-2)(x-7) &= 0
\end{aligned}$$

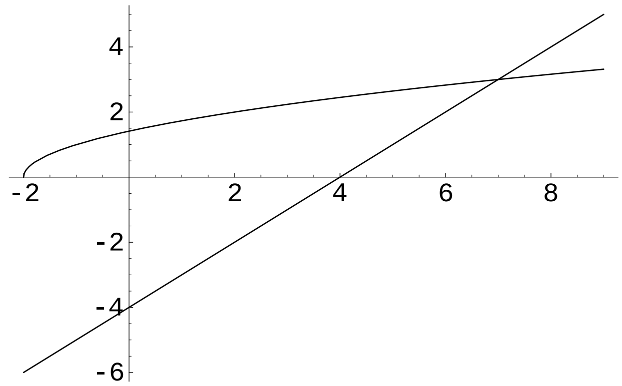
So, x is either 2 or 7. This isn't the end, however. We should check both answers in the original equation. We find that $x = 7$ works, but $x = 2$ does not. Our solution is $x = 7$.

Why did we have to check our answers? I mean, it's always a good idea, but in this case it was more than that—it was necessary! The reason is that when we squared both sides, we destroyed a little information. How? Consider, for instance: we know that 3 and -3 are not the same, but when we square them both, we get 9. That is, by squaring, we have lost the ability to distinguish between them—we have lost information. Let's follow the extraneous solution $x = 2$ through the algebra above to see how we lost the information.

Original algebra	With $x = 2$	Simplified	Remarks
$\sqrt{x+2} = x-4$	$\sqrt{2+2} = 2-4$	$2 = -2$	This is false.
$x+2 = (x-4)^2$	$2+2 = (2-4)^2$	$4 = (-2)^2$	This is true.
...			

We know that $x = 2$ is wrong by looking at the top line. After all, 2 and -2 are different. However, when we square both sides, they both turn into 4—and we can no longer tell the difference between them. This is how the false solution $x = 2$ snuck into consideration, and why we must check solutions in the original equation.

(b) Graph the functions $y = x - 4$ and $y = \sqrt{x+2}$ on the same axes. What does your graph have to do with your answer to part (a)?



The point (7,3) is on both graphs. If we had included the bottom half of the parabola (which is $y = -\sqrt{x+2}$), then there would be another intersection at (2,-2). But we didn't, and there isn't.

Problem 11 Calculate $(\sqrt{5}-1)(\sqrt{5}+1)$.

$(\sqrt{5}-1)(\sqrt{5}+1) = 5 - \sqrt{5} + \sqrt{5} - 1 = 4$

Problem 12 **(a)** Find the midpoint of (2,5) and (26,15).

$$\left(\frac{2+26}{2}, \frac{5+15}{2}\right) = \left(\frac{28}{2}, \frac{20}{2}\right) = (14, 10).$$

(b) How far is the midpoint from each of the endpoints?

$d = \sqrt{(14-2)^2 + (10-5)^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13.$

(c) Write the equation of the circle whose diameter has endpoints (2,5) and (26,15).

$(x-14)^2 + (y-10)^2 = 169.$

Problem 13 **(a)** *How far is the point* $(1, \sqrt{3})$ *from the origin?*
Distance from $(1, \sqrt{3})$ to (0,0):
$d = \sqrt{(1-0)^2 + (\sqrt{3}-0)^2} = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2.$

(b) Given that the point $(1, \sqrt{3})$ lies on a circle centered at the origin, what is the radius of that circle?

Same as the distance to the origin: 2.

(c) What is the slope of the line segment from the origin to the point $(1, \sqrt{3})$?

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\sqrt{3} - 0}{1 - 0} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

(d) What equation describes the line tangent to the given circle at the point $(1, \sqrt{3})$?

A tangent line to a circle is perpendicular to the radius there. So, the line tangent to the circle at $(1, \sqrt{3})$ is perpendicular to the line through (0,0) and $(1, \sqrt{3})$. Perpendicular slope is $-1/m$, which in this case is $-\frac{1}{\sqrt{3}}$. So, we have:
$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 1)$.
That's enough there, or we could simplify:

$$\begin{aligned} y - \sqrt{3} &= -\frac{1}{\sqrt{3}}(x - 1) \\ y - \sqrt{3} &= -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \\ y &= -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} + \sqrt{3} \end{aligned}$$

Problem 14 You wrote the equation of a certain circle on your hand right before playing racquetball, and now one of the numbers is illegible. What remains of the equation is: $x^2 - 6x + y^2 - 8y + \clubsuit = 0$. Can you still find the center of the circle? (Here $\clubsuit$ is the illegible smudge.)

Completing the square, we have

$$\begin{aligned} x^2 - 6x \quad + y^2 - 8y \quad + \clubsuit &= 0 \\ x^2 - 6x + 9 + y^2 - 8y + 16 + \clubsuit &= 9 + 16 \\ (x-3)^2 + (y-4)^2 &= 25 - \clubsuit \end{aligned}$$

From this equation, we cannot determine the radius, because the right-hand side is messy. But from the left-hand side, we see that the center is at (3,4).

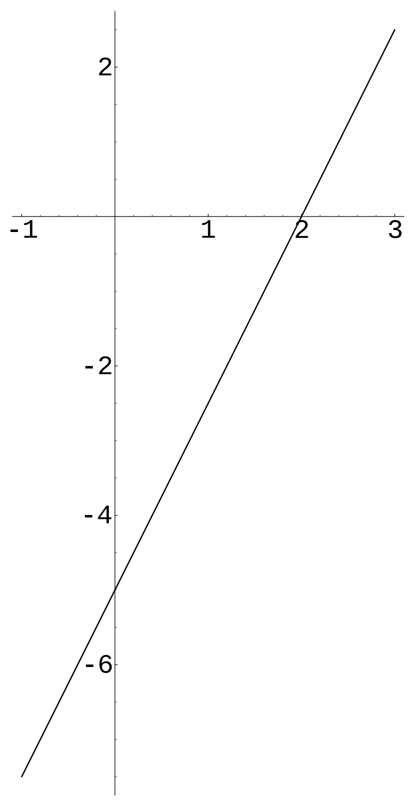
Problem 15 Find the value of k for which the graph of $y = 2(x-1)^2 + k$ passes through the origin.

Set $x = y = 0$. We have:

$$\begin{aligned} 0 &= 2(0-1)^2 + k \\ 0 &= 2 + k \\ k &= -2 \end{aligned}$$

Problem 16 Find the x- and y-intercepts of the line $5x - 2y = 10$, and plot the line.

For $x = 0$, we have $0 - 2y = 10$, so $y = -5$.
For $y = 0$, we have $5x - 0 = 10$, so $x = 2$.



Problem 17 Let $f(x) = \frac{|x|}{x}$ for all $x \neq 0$. Evaluate $f(x)$ for $x = \pm1, \pm2, \pm3$. Graph $f(x)$.

$f(x) = 1$ for $x > 0$, and $f(x) = -1$ for $x < 0$. That's two horizontal rays, one pointing to the right from (0,1) and the other pointing left from (0,-1).

Problem 18 **(a)** Write an equation for the horizontal line with y-intercept 3. Is this equation in slope-intercept form?
$y = 3$, yes.

(b) Write an equation for the vertical line with x-intercept 1. Can you put this equation in slope-intercept form?

$x = 1$, no. There is no slope.

Problem 19 Write an equation for the line with y-intercept 3 and x-intercept 4. Put this equation in the form $ax + by = c$. Now divide by c to get an equation in the form $Ax + By = 1$. Do you notice anything interesting?

This line passes through the points (0,3) and (4,0). Therefore, its slope is $m = -3/4$, so we have $y = (-3/4)x+3$. That is, $\frac{3}{4}x+y = 3$, or (dividing by 3) $\frac{x}{4}+\frac{y}{3} = 1$. Note how the denominators are the intercepts.

Problem 20 Consider the function $y = x^2$. Two points on it are $A(a, a^2)$ and $B(b, b^2)$. Find the slope of segment AB.

$$m = \frac{a^2 - b^2}{a - b} = \frac{(a-b)(a+b)}{(a-b)} = a + b.$$

Problem 21 What is the average rate of change of $y = x^3 - 1$ on the interval [-1,2]?

The graph goes through (-1,-2) and (2,7). So, slope is $m = \frac{7-(-2)}{2-(-1)} = \frac{9}{3} = 3$.

Problem 22 Without graphing it, determine whether the graph of $x^2+y^4-3y^2 = 1$ is symmetric with respect to the x-axis, the y-axis, and/or the origin.

Changing the signs on either x or y has no effect on this equation, so it is symmetric about all three.

Problem 23 **(a)** For what value of x is the value of $f(x) = (x+1)^2 + 3$ as small as possible?

Setting $x = -1$ makes the squared term vanish.

(b) What is the minimal value of f?

$f(-1) = 3$.

(c) What is the range of f?

From 3 on up, i.e. $[3, \infty)$.

Problem 24 Graph $y = 2 - |x + 1|$. What are the coordinates of its vertex?

(-1,2) is the vertex. The graph is a downward pointing V from there.

Problem 25 Find the domain, range, horizontal asymptote, and vertical asymptote of $y = \frac{2x+4}{x-1}$.

Domain: All reals except 1 (because $x = 1$ would have us dividing by zero).
Range: All reals except 2. This is a little tricky to see; one way is to solve the equation for x:

$$\begin{aligned} y &= \frac{2x+4}{x-1} \\ (x-1)y &= 2x+4 \\ xy - y &= 2x+4 \\ xy - 2x &= y+4 \\ x(y-2) &= y+4 \\ x &= \frac{y+4}{y-2} \end{aligned}$$

That is, y could have any value except 2.
Vertical asymptote: $x = 1$. When $x = 1$, the denominator of the original equation is zero, but the numerator is not.
Horizontal asymptote: $y = 2$. Some ways to see this:

$$\begin{aligned} y &= \frac{2x+4}{x-1} \\ &= \frac{2x-2+6}{x-1} \\ &= \frac{2x-2}{x-1} + \frac{6}{x-1} \\ &= 2\frac{x-1}{x-1} + \frac{6}{x-1} \\ y &= 2 + \frac{6}{x-1} \end{aligned}$$

For large values of x, the fractional part becomes very small, so the whole expression gets very close to the horizontal asymptote $y = 2$. Or...

$$y = \frac{2x+4}{x-1} = \frac{x(2+\frac{4}{x})}{x(1-\frac{1}{x})} = \frac{2+\frac{4}{x}}{1-\frac{1}{x}} \to \frac{2}{1} = 2$$

At the step in which the fractions $\frac{4}{x}$ and $\frac{1}{x}$ magically disappear, the key observation is that for large values of x, these fractions are so small we may ignore them.