The Linear Combination and Friends.

I don't have much of an introduction for you—but I can say that I have enjoyed all of the 210's I've taken, so I'm very excited to have the chance to teach one. This quarter we'll be talking about convex sets and some related ideas. If we have time, we'll look into affine transformations, but I would rather cover a little material well than a lot of material poorly. Most of the material for this course comes from Frederick Valentine's Convex Sets, Steven Lay's Convex Sets and Their Applications, and David Barnette's The Geometry of Convex Sets. Let's get started.

We begin in E^n, the n-dimensional Euclidean space. This is the set of all real n-tuples $(a_1, a_2, \ldots, a_n)$, with three operations (defined for all real numbers λ, a_i, and b_i, where $1 \leq i \leq n$):

- Vector Addition: $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
- Scalar Multiplication: $\lambda(a_1, a_2, \ldots, a_n) = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$
- Euclidean Inner Product (a.k.a. Dot Product):

$$(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = \sum_{i=1}^{n} a_i b_i.$$

We will call members of E^n points or vectors. To distinguish the zero vector from the number zero, we use the boldface $\mathbf{0}$ to represent the vector. At this moment, I'm not especially worried about distinguishing between the zero vector in E^m and the zero vector in E^n. If circumstances necessitate, we can use the more specific labels $\mathbf{0}^m$ and $\mathbf{0}^n$. Unless otherwise specified, we will always assume that we are working in the context of E^n.

The beauty of an electronic document is that if I later decide we needed more terminology here in the front, I can add it later. Woo hoo!

Given a collection of points $x_1, \ldots, x_k$ and real numbers $\alpha_1, \ldots, \alpha_k$, we may form the linear combination $\alpha_1 x_1 + \ldots + \alpha_k x_k = \sum_{i=1}^{k} \alpha_i x_i$.

If we add the constraint that $\sum \alpha_i = 1$, then we call this an affine combination.

Alternatively, we could require that $\alpha_i \geq 0$ for all i and $\sum \alpha_i > 0$. In this case, we call it a positive combination.

A positive combination that is also an affine combination is called a convex combination. That is, a convex combination is a linear combination in which the coefficients are nonnegative and sum to 1.

Let's illuminate this with an example of two points in the plane E^2: say, $x = (1, 2)$ and $y = (2, 1)$. Since these vectors are linearly independent (and there are two of them), linear algebra tells us that they span the plane. Thus, the set of all linear combinations of x and y is, in fact, all of E^2. (For your own amusement, you could produce an argument on basic principles—one that doesn't rely on sophisticated results of linear algebra.)

Now we consider the set of affine combinations of x and y. If u is an affine combination of x and y, then there exist real numbers α and β such that $u = \alpha x + \beta y$ and $\alpha + \beta = 1$. We would do well to let $t = \alpha$, so that $\beta = 1 - t$. Now $u = t(1,2)+(1-t)(2,1) = (t, 2t)+(2-2t, 1-t) = (2-t, t+1) = (2,1)+t(-1,1)$. You might recognize this last expression as the vector form of the equation of the line through (1,2) and (2,1), so we have shown that an affine combination of x and y lies on the line through x and y. Furthermore, given any point on that line, we could quickly recover the corresponding value of t, and therefore the values of α and β needed to express that point as an affine combination of x and y. Thus, the line containing x and y is precisely the set of all affine combinations of x and y.

Before we look at the positive combinations of x and y, let's see their convex combinations. They will be all points u of the form $u = t(1, 2) + (1 - t)(2, 1)$, with the added constraint that $t \geq 0$ and $1 - t \geq 0$. Equivalently, the convex combinations of (1,2) and (2,1) are all points $u = (2, 1) + t(-1, 1)$ where $0 \leq t \leq 1$. You may confirm that this gives the closed line segment from (1,2) to (2,1).

Now we turn our attention to the positive combinations. Suppose $u = \alpha x + \beta y$ is a positive combination of x and y. This is to say that $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta > 0$. In other words, neither of the coefficients is negative and they aren't both zero. If $c = \alpha + \beta$, then $\frac{\alpha}{c} \geq 0$, $\frac{\beta}{c} \geq 0$, and $\frac{\alpha}{c} + \frac{\beta}{c} = 1$. We let

$$v = \frac{u}{c} = \frac{\alpha}{c}x + \frac{\beta}{c}y,$$

and observe that v is a convex combination of x and y—so v is a point on segment $\overline{xy}$. We also have $u = cv$. This may sound like a cynical axiom of the academic job hunt[1], but actually it says that u is a positive multiple of v. This means that u lies on the open ray $\overrightarrow{\mathbf{0}v}$. We conclude that u lies on an open ray from the origin to some point on segment $\overline{xy}$. These rays, taken together, form a <u>pencil</u>. Another way to say that is: u lies on or in $\angle x\mathbf{0}y$, but not at its vertex, $\mathbf{0}$. (I see two other ways of visualizing this set of positive combinations. Visualization 1: rather than

[1]Sorry. Bad joke.

fire rays through $\overline{xy}$, we can instead consider the family of positive scalar multiples of this segment, i.e. $c(\overline{xy}) = \{cv : v \in \overline{xy}\}$. For example, $2(\overline{xy})$ is the segment from (2,4) to (4,2). As c ranges through the positive reals, we get a family of parallel segments that shrink towards **0** or stretch into the distant reaches of the first quadrant: their left-hand endpoints are always on the open ray from **0** through (1,2), their right-hand endpoints are always on the open ray from **0** through (2,1), and their slopes are all -1. When we collect together all the points that lie on some such segment, we get the union of a ray's–worth of segments. Previously, we were looking at the union of a segment's–worth of rays. Visualization 2: by a change of basis, we can correspond any linear combination $\alpha x + \beta y$ with the vector (α, β). In the case of positive combinations, these resulting vectors lie in the closed first quadrant of E^2 (minus the origin). Undoing the change of basis, this first quadrant folds, somewhat like an umbrella, to give us the wedge—or rather, *pencil*—described earlier.)

We will often find that a theorem or definition involving linear combinations is just as valid if the word "linear" is replaced throughout by one of the words, "affine", "positive", "convex". Rather than write the same thing four times, we'll use an asterisk to indicate which words can be replaced. For example, consider this definition:

A set S is called <u>linear*</u> if it is closed under pairwise linear* combination.

The asterisk shorthand indicates that we've given four definitions here. One of them is:

A set S is called <u>convex</u> if it is closed under pairwise convex combination. That is, S is a convex set if for any pair of points $x, y \in S$ and any real numbers $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $\alpha x + \beta y \in S$.

Let's give these definitions a little geometric significance. We're saying that S is:

- linear, if it contains[2] the span of any two of its members
- affine, if it contains the line through any two of its (distinct) members
- convex, if it contains the line segment between any two of its members
- positive, if for any $x, y \in S$, S also contains $\angle x\mathbf{0}y$ and its interior, but perhaps not **0** itself.

[2] Contains as a subset, not as an element.

The geometric meaning of being a positive set is a little messy, and not totally described by the above. We've already seen an example where x and y are chosen in such a way that the line through them does not include $\mathbf{0}$, but you may find it helpful to consider several cases in which $\mathbf{0}$ is colinear with x and y. Perhaps $\mathbf{0}$ is between x and y; perhaps it is one of them; perhaps x is between $\mathbf{0}$ and y.

Exercise 1 Let $x_1 = \mathbf{0}$, $x_2 = (1,1)$, $x_3 = (-1,1)$, $x_4 = (0,-1)$. Draw/describe the sets of all linear* combinations of:

1. x_1 and x_2
2. x_2 and $-x_2$
3. x_2, x_3, and x_4
4. x_1, x_2, x_3, and x_4

By the way, we'll soon prove that the word "pairwise" in the definition of a linear* set is not very important: if a given set contains every linear* combination of any *two* of its elements, then it also contains every linear* combination of *any finite number* of its elements. But we have other stuff to do first.

Suppose S is a linear set. It follows that for any x, $y \in S$, every linear combination of x and y also belongs to S. Because every convex combination is a linear combination, this implies that every convex combination of x and y belongs to S; thus, S is a convex set. Similarly, every linear set is positive and affine, and every affine or positive set is convex.

Exercise 2 Every positive set is convex. Find an example of a convex set that is not positive.

Theorem 1 A nonempty set S is linear if and only if it is affine and contains $\mathbf{0}$.

Proof: As we just saw, a linear set must be affine. If S is a nonempty linear set, then it contains some point x. Now $\mathbf{0} = 0x + 0x$, which is a linear combination of two elements of S; thus, since S is linear, $\mathbf{0} \in S$. Therefore we have shown that a nonempty linear set is affine and contains $\mathbf{0}$.

For the reverse, we suppose S is an affine set containing $\mathbf{0}$. To prove that S is a linear set, we suppose that x and y are members of S, and consider the arbitrary linear combination $\alpha x + \beta y$. We intend to show that this belongs to S. Note that x, y, and $\mathbf{0}$ are members of S. Thus, $z_1 \equiv 2\alpha x + (1-2\alpha)\mathbf{0}$ and $z_2 \equiv 2\beta y + (1-2\beta)\mathbf{0}$

belong to S as well, since each of them is an affine combination of two members of S. Therefore, S also contains any affine combination of z_1 and z_2; in particular, it contains

$$\frac{1}{2}z_1 + \frac{1}{2}z_2 = \frac{1}{2}[2\alpha x + (1-2\alpha)\mathbf{0}] + \frac{1}{2}[2\beta y + (1-2\beta)\mathbf{0}] = \alpha x + \beta y.$$

And we're done.

Exercise 3 Prove that a set S is positive if and only if it is convex and for every $x \in S$, S contains the open ray from $\mathbf{0}$ through x.

Theorem 2 A set S is linear if and only if it is affine and positive.

Proof: The forward implication follows from the discussion preceding Theorem 1. For the reverse, we first note that the claim is vacuously true if S is empty, so suppose S is an affine, positive set containing some point, x. Since S is a positive set, S contains $1x+1x = 2x$. Since S is affine and 2-1=1, it follows that S contains $2(x) - 1(2x) = \mathbf{0}$. Now S is an affine set containing $\mathbf{0}$, so S is linear by Theorem 1.

As Theorem 1 quietly suggests, whether a set is linear or not depends on where that set is. So, now it's time to start moving sets around. If A is a set and x is a point, then $A + x \equiv \{a + x : a \in A\}$. The set $A + x$ is called a translate of A. We may analogously define, for sets A and B, $A + B \equiv \{a + b : a \in A,\ b \in B\}$. We also define the scalar multiples of a set: for any real number λ and set A, we say $\lambda A = \{\lambda a : a \in A\}$. If B is a translate of a scalar multiple of A—that is, if $B = \lambda A + x$ for some real number λ and point x—then we say B is homothetic to A. (That's not a word I see every day, so I wanted to get it out on the table while I could.)

Exercise 4 Prove or give a counterexample for each of the four claims: Any translate of a linear* set is linear*.

Exercise 5 Let $S = \{(0,0),(1,0)\}$. Find convex sets A and B such that $A + S$ is convex and $B + S$ is not. Find linear sets C and D such that $C + S$ is linear and $D + S$ is not.

Exercise 6 Prove or disprove: If A and B are linear*, then so is $A + B$.

Exercise 7 Prove or disprove: If A is linear* and B is homothetic to A, then B is linear*.

Exercise 8 Prove that if $\lambda \neq 0$ and A is linear, then $\lambda A = A$.

Exercise 9 Prove that $\lambda A + \lambda B = \lambda(A + B)$.

Exercise 10 Prove that if $\lambda \neq 0$ and A is affine, then λA is a translate of A.

Exercise 11 Prove that if two translates of an affine set intersect, then they are identical.