Notes from April 5, 2005. (I may sandwich all this stuff into a single file later, which would resolve some numbering issues. But for now, I hope that making the files separate will at least keep load times down.)

We say that a set S is star shaped relative to a point $x \in S$ if and only if, for every point $y \in S$, it is true that $\overline{xy} \subseteq S$. (Some authors would say that such a set S is "star convex" with respect to x, reserving the term "star shaped" for sets that are star convex but not convex. But I'm not those authors.)

This is related to the Art Gallery Problem: given a polygonal art gallery, what is the minimal number of cameras needed to guard it? For more detail, visit `http://cgm.cs.mcgill.ca/~godfried/teaching/cg-projects/...`
`...97/Thierry/thierry507webprj/artgallery.html` .
To say that a set is star shaped relative to x is to say that a single camera at point x could watch the whole gallery. We might ask: where could such a camera be placed? That is, what is the set of all points $x \in S$ relative to which S is star shaped? Even if we don't have an answer, we do have a name for the answer: *kernel*.

The kernel of S is the set of all $x \in S$ such that S is star shaped relative to x. It is denoted $\ker S$.

Exercise 1 Find $\ker S$, where:

- S is convex
- S is bounded by four quarter-circles that bend inward instead of outward
- S is the Carl's Jr. star
- S is a crescent (and how do you define "crescent"?)

Exercise 2 Find a set S whose kernel is nonempty and lies in the interior of S.

Exercise 3 Find a polygon of as few sides as possible whose kernel is:

- a point
- a line segment
- empty

Exercise 4 How circle-like can a set be and still have empty kernel?

Exercise 5 †Let K be a convex set. Under what conditions will there exist a nonconvex set S whose kernel is K?

Since a set is convex if and only if it contains $\overline{xy}$ for all $x, y \in S$, there is a connection between these definitions: a set is convex if and only if it is star shaped relative to each of its members. Equivalently, S is convex if and only if $S = \ker S$. This implies that the kernel of a convex set is convex, but this result also holds in general:

Theorem 1 For any set S, $\ker S$ is convex.

Proof: Let $K = \ker S$. We must show that K is convex. Let's rewrite this claim in a sequence of equivalent forms until we reach one that can be proven easily:

1. For any x, y in K, $\overline{xy} \subseteq K$.
2. For any x, y in K, and for any $u \in \overline{xy}$, $u \in K$.
3. For any x, y in K, and for any $u \in \overline{xy}$, it follows that for any $z \in S$, $\overline{uz} \subseteq S$.
4. For any $x \in K$, $y \in K$, $u \in \overline{xy}$, $z \in S$, and $v \in \overline{uz}$, we have $v \in S$.

We'll prove the last of the above statements. Let x and y belong to K. Let $u \in \overline{xy}$; then $u = \alpha x + (1-\alpha)y$ for some $\alpha \in [0,1]$. Let $z \in S$ and $v \in \overline{uz}$; then $v = \beta u + (1-\beta)z$ for some $\beta \in [0,1]$. That is:

$$\begin{aligned} v &= \beta u + (1-\beta)z \\ &= \beta[\alpha x + (1-\alpha)y] + (1-\beta)z \\ &= \alpha\beta x + (\beta - \alpha\beta)y + (1-\beta)z \\ &= \alpha\beta x + (1-\alpha\beta)\left[\frac{\beta-\alpha\beta}{1-\alpha\beta}y + \frac{1-\beta}{1-\alpha\beta}z\right] \\ v &= \alpha\beta x + (1-\alpha\beta)w, \end{aligned}$$

where w is the mess in square brackets. Note that w is a convex combination of y and z, because $\frac{\beta-\alpha\beta}{1-\alpha\beta} \geq 0$, $\frac{1-\beta}{1-\alpha\beta} \geq 0$, and $\frac{\beta-\alpha\beta}{1-\alpha\beta} + \frac{1-\beta}{1-\alpha\beta} = \frac{1-\alpha\beta}{1-\alpha\beta} = 1$. Since $y \in K$ and $z \in S$, this implies that $w \in S$, by the definition of kernel. We now have that v is a convex combination of x and w, so by similar logic, $v \in S$. Thus $u \in K$ and K is convex.

We now define the handy and standard (standy?) *hulls*. We've just shown that "taking the kernel" is one way to get from any set to a uniquely determined convex subset. Now let's go the other direction: find a way of building a convex superset of S. Without further adieu:

The convex* hull of S is the intersection of all convex* sets containing S. The convex hull of S is denoted con$S$; the affine hull is aff$S$; the positive hull is pos$S$; the linear hull is lin$S$.

The linear hull is more commonly recognized as the *span*, from linear algebra.

Exercise 6 Prove that con^*S is convex*.

It follows that con^*S is the smallest convex* set containing S.

Exercise 7 Give an example of a set S such that "the largest convex subset of S" is not uniquely determined.

By defining the hulls as intersections of families of sets, we've essentially given a way to construct a hull of S from the outside—that is, by whittling away excess material from the space in which S lives. We can also devise an internal construction—a way to start with S and add only what we need in order to build its hull (from within). We might guess that the convex* hull of S is the union of all convex* subsets of S.

Exercise 8 For a variety of sets S, find the union of convex* subsets of S.

As you'll quickly notice, this isn't working out. Next time, we'll see a correct internal construction for the convex* hull of S.