Last time, we defined the convex* hull of a set $S \subseteq E^n$ by way of an external construction, by which I mean that we took the entire space E^n and whittled it down as much as possible without throwing away any members of S. That is, we formed

$$\operatorname{con}^* S = \bigcap_{R \in \mathcal{C}} R,$$

where $\mathcal{C}$ is the set of all convex* sets in E^n that contain S.

Now we'll give an alternate construction, building the same set up from within. For any natural number k, define

$$\operatorname{con}^*_k S = \left\{ \sum_{i=1}^{k} \alpha_i x_i : x_i \in S \text{ and } \{\alpha_i\}_{i=1}^{k} \text{ satisfy convex* conditions} \right\}.$$

That is, $\operatorname{con}^*_k S$ is the set of all convex* combinations of k or fewer members of S. Note that $S \subseteq \operatorname{con}^*_1 S$, with equality in the convex and affine cases; furthermore, if $a < b$, then $\operatorname{con}^*_a S \subseteq \operatorname{con}^*_b S$.

Exercise 1 Provide an example of a set S and a number $k > 1$ such that $\operatorname{con}_k S$ is not convex.

Exercise 2 Let $S = \{(1,0), (0,1), (-1,-1)\}$. Find:

- $\operatorname{pos}_1 S$
- $\operatorname{con}_2(\operatorname{pos}_1 S)$
- $\operatorname{pos}_1(\operatorname{con}_2 S)$
- $\operatorname{lin}_2 S$

Let X be the set of all convex* combinations of members of S. In other words,

$$X = \bigcup_{i=1}^{\infty} \operatorname{con}^*_i S.$$

Exercise 3 Prove that X is convex*.

Thus X is one of the sets in the intersection that defines $\operatorname{con}^* S$, so $\operatorname{con}^* S \subseteq X$.

Exercise 4 Prove that for any natural number k, $\operatorname{con}^*_k S \subseteq \operatorname{con}^* S$.

Thus $X \subseteq \operatorname{con}^* S$, so in fact the two sets are equal.

Exercise 5 Prove that for convex sets X and Y, $\mathrm{con}(X \cup Y) = \{\alpha x + \beta y : x \in X, y \in Y, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\}$. That is, the convex hull of $X \cup Y$ is the union of all segments that start in X and end in Y. Formulate a similar statement for affine sets X and Y. Is it true?

Exercise 6 Let $S = \{(-1, 1), (1, 1)\}$. Find $\mathrm{pos}S$, and note that $\mathrm{pos}S \cap \mathrm{aff}S = \mathrm{con}S$.

We might think that, in general, $\mathrm{pos}S \cap \mathrm{aff}S = \mathrm{con}S$.

Exercise 7 Check the above claim for validity on the set $S' = \{(-1, 1), (1, 1), (1, 2)\}$. What's the difference between S and S'?

Now we introduce variations on the idea of linear independence. Recall from linear algebra that a set of vectors is linearly dependent if one of them can be written as a linear combination of the others. You may be familiar with a different definition: a set of vectors is linearly dependent if some nontrivial linear combination of its members is the zero vector. It is not hard to show that these definitions are equivalent for linear dependence; however, the latter doesn't generalize very nicely. Here is the version we will use:

A point x is <u>linearly* dependent on a set S</u> if $x \in \mathrm{lin}^* S$. Otherwise it is <u>linearly* independent of S</u>. A set S is called <u>linearly* independent</u> if each of its members is linearly* independent of the others; otherwise we call S <u>linearly* dependent</u>.

Exercise 8 Which of the four types of independence are invariant under translation? That is: for what choice of "*" is the set B convexly* independent if and only if $B + v$ is, too? Prove your answers.

Recall that a linearly independent set of points that span a subspace V were called a *basis* for V in linear algebra. We would like the same definition, but we must be a little more specific (since we are considering so many special cases of the linear combination).

We say that a <u>convex* basis</u> of a set S is a set B such that $\mathrm{con}^* B = S$. Clearly this requires that S be a convex* set.

In linear algebra, we learned that a linear basis of E^n always has exactly n points. Something similar happens for affine combinations, as we will soon see.

Exercise 9 Prove that $\mathrm{aff}(B + v) = (\mathrm{aff}B) + v$.

Exercise 10 Prove that if $\mathbf{0} \in B$ and $B \neq \{\mathbf{0}\}$, then $\text{aff}B = \text{lin}(B \backslash \{\mathbf{0}\})$.

Exercise 11 Prove that an affine basis for E^n must have exactly $n+1$ elements.

Another useful property of a linear basis B of a set S is that any point in S has a unique representation as a linear combination of points in B.

Exercise 12 Prove that if B is an affine basis for S, then every $x \in S$ has a unique representation as an affine combination of points in B.

Exercise 13 If $B = \{x_1, \ldots, x_k\}$ is an affine basis for S, and $x = \sum \alpha_i x_i$ is an affine combination of the points in B, then the coefficients α_i are the <u>affine coordinates</u> of x with respect to B. With $B = \{(4, 0), (2, 2), (-1, 1)\}$, find the affine coordinates of (4,6).

However, similar properties do not hold for positive and convex sets.

Exercise 14 Find a positive basis for E^n comprising $n + 1$ points; find another comprising $2n$ points.

Exercise 15 Find a positive basis for E^2 with respect to which every point in the plane has more than one representation.

The situation is far stranger for convex sets:

Exercise 16 Find a convex set S with an infinite convex basis.

Exercise 17 Prove that E^1 has no convex basis.