This is intended to clear up some of the confusion about **centroids.**

We should start with the idea of a region R and a line L, both in the plane. (Check out the pictures in Stein, p.494.) Imagine that L is a tightrope stretched across a chasm of some sort, and R is trying to scoot across without falling off the rope. Of course this sounds very dangerous, so there's a vulture soaring overhead, waiting for lunch. When we look at a picture of the plane with R and L in it, we are seeing through the vulture's eyes.

Let's talk about the *moment of* R *about* L, which is a very important number. We can call it M_L for short. This describes the tendency of R to roll off the rope. That is, if $M_L = 0$, then R is balanced and won't fall off. Otherwise, it's lunch time for the vulture (who probably deserves a name by now; let's call it V). Note that in fig. 6, it doesn't look like R is even touching L. That's very bad news for R, and rather good news for V. It seems like R could at least have a *chance* with line L' in fig. 7.

Okay, how should we compute M_L? You may remember some discussion of levers and such; for a reminder, see pp.492-3. We slice R into a bunch of little pieces (don't worry, it's painless) parallel to L. If we change perspective to look at R from one end of L, we will see L as a dot and R as a bunch of short line segments which used to be one whole line segment. The incisions we made on R appear as the endpoints of all those little line segments. Now we have something that looks very much like the lever problem: a fulcrum (the "point" which is really the end-on view of L) and a bunch of small masses (the short line segments) distributed across a beam. We just want to add up a bunch of quantities that look like ($lever\ arm \cdot mass$). To make the numbers simpler, say L is the y-axis for a minute; this gives us

$$M_L = M_y = \int_a^b lever\ arm \cdot mass\, dx = \int_a^b x \cdot f(x)\, dx$$

where $f(x)$ is the length of the slice of R with the appropriate x-coordinate. You've seen this before. So let's skip ahead just a bit and try to figure out how to derive the centroid formula.

The *center of mass* of R is a special point with the property that R will balance on any line through that point. If R has uniform density, then the center of mass is the same as the *centroid* of R, which you could think of as the "center of volume". Centroid is a spacial concept; center of mass involves mass *and* space.

So let's assume that R has uniform density. You can show that it doesn't matter what the density is, as long as it is constant; for simplicity, let's call it 1. Let's say R has centroid $(\overline{x}, \overline{y})$. We don't know the actual numbers that go there, but we can work with $\overline{x}$ and $\overline{y}$ a little, to figure them out. We know that R will balance on any line through its centroid, including the line $x = \overline{x}$. Since R balances on that line, which we may as well call L'', we know that the moment of R about L'' is 0. (As in: V will get 0 lunch today.) That is:

$$\begin{aligned}
\int_a^b (x - \overline{x}) \cdot f(x)\, dx &= 0 \\
\int_a^b x \cdot f(x)\, dx - \int_a^b \overline{x} \cdot f(x)\, dx &= 0 \\
\int_a^b x \cdot f(x)\, dx &= \int_a^b \overline{x} \cdot f(x)\, dx \\
\int_a^b x \cdot f(x)\, dx &= \overline{x} \cdot \int_a^b f(x)\, dx \\
\overline{x} &= \frac{\int_a^b x \cdot f(x)\, dx}{\int_a^b f(x)\, dx}
\end{aligned}$$

Note that $(x - \overline{x})$ is the lever arm. We can make a similar computation for $\overline{y}$, so we have derived the centroid formula.