A sample $\delta - \epsilon$ proof related to the material in Section 14.4:

$$\text{Prove that } \lim_{(x,y)\to(3,1)} 2x + y = 7.$$

Forethought:

We have a function which takes two-dimensional input from the xy-plane, and produces one-dimensional output. Given a tolerance ϵ along the z-axis, we need to find a radius δ such that the disc (in the xy-plane) of radius δ, centered at (3,1), is mapped into the interval (on the z-axis) of radius ϵ, centered at 7.

In other words, for any $\epsilon > 0$ we must be able to exhibit a $\delta > 0$, such that whenever (x, y) is within δ units of the point (3,1), $f(x, y)$ is within ϵ units of 7. This way of thinking may appear complicated and strange, but it's the most reasonable way to define continuity for now. There is another, more elegant way to define it, but it requires knowledge of set theory. Places you might find the "topological" definition of continuity are Math 108, Math 127, and Math 147. If you can produce your own definition of continuity, I would love to see it. Thinking about this should help you understand the definition we have.

Work:

Now we'll start with what we have, and work towards what we need. We are given ϵ, and must find δ.

By restricting δ, we want to force $|(2x + y) - 7|$ to be less than ϵ. Starting with the inequality $|2x + y - 7| < \epsilon$, we'll work towards some sufficient conditions on δ.

$$\begin{aligned} |2x + y - 7| &< \epsilon \\ |2x - 6 + y - 1| &< \epsilon \\ |2(x - 3) + (y - 1)| &< \epsilon \end{aligned}$$

If we impose the condition that (x, y) be within δ units of (3,1), we are forcing points to lie within a disc of radius δ, centered at (3,1). That disc lies within a square of side length 2δ, and on that square, we can say that $|x - 3| < \delta$ and $|y - 1| < \delta$. So,

$$\begin{aligned} |x - 3| &< \delta \\ 2|x - 3| &< 2\delta \\ |y - 1| &< \delta \\ |2(x - 3) + (y - 1)| &\le |2(x - 3)| + |(y - 1)| \\ |2(x - 3) + (y - 1)| &< 2\delta + \delta \\ |2(x - 3) + (y - 1)| &< 3\delta \end{aligned}$$

Again, we are trying to find a value of δ which forces $|2x + y - 7| < \epsilon$. So, we should pick δ so that 3δ is *at least as restrictive as* ϵ. So, we choose $\delta = \epsilon/3$. We could also have chosen anything smaller, such as $\delta = \epsilon/4$.

Reverse:

Now, we will use our work to construct a proof. You might say that our search for δ was working backwards, so now we organize the process into forward-flowing logic.

For any given $\epsilon > 0$, we choose $\delta = \epsilon/3$. Now, suppose that the point (x, y) lies inside the disc (in the xy-plane) of radius δ, centered at (3,1). Then we can be sure that x is within δ of 3 and y is within δ of 1. Therefore:

$$
\begin{aligned}
|x-3| &< \delta \\
|y-1| &< \delta \\
2|x-3| + |y-1| &< 2\delta + \delta \\
2|x-3| + |y-1| &< 3\delta \\
|2(x-3) + (y-1)| &< 3\delta \quad \leftarrow \text{Triangle Inequality} \\
|2x - 6 + y - 1| &< 3\delta \\
|2x + y - 7| &< \epsilon
\end{aligned}
$$

So, when we assume that (x, y) is within δ of (3,1), it follows as a logical consequence that $2x + y$ is within ϵ of 7. This satisfies the definition of the limit, so the result is proven.

Remark: We have actually shown not only that the limit is 7, but also that the function is continuous there. In order to show *just* that the limit is 7, we should ignore the behavior of the function at (3,1) itself. However, since this function was continuous, we had no need to do so.

Remark 2: The Triangle Inequality, mentioned above, comes from the fact that on any triangle, no side can be longer than the sum of the other two sides. It is often written

$$|A + B| \leq |A| + |B|.$$

Although the name makes the most sense when you think in terms of vectors, we can see that this is true in the case of numbers because, if A and B have different signs, there will be some cancellation when we add them together. This will result in a smaller overall number than if we take their absolute values first. Check this out with a few real numbers of your own choosing. Try these cases: both numbers positive, both negative, one positive with the other negative, one zero.