The condensed version of the precise definition of

$$\lim_{x \to c} f(x) = L$$

is the crazy scribble

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \ni [0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon] \,.$$

From the moment I first saw this line of impenetrable nonsense, it was love. I was sure I had stumbled upon the Code of All Things, the secret language of People In The Know. I committed this definition to memory because it was *so* alien, it was beautiful. Did I mention I enjoy the works of H.R. Giger?

Over the last decade, my attraction to that single line of symbols has changed: what was once some form of idolatry has become a healthier respect for its *information density.* Let me explain. Now and then, writers of fiction like to create small pieces of food with incredible nutritional value: concentrated meals. A couple examples that come to mind are elven lembas bread from Lord of the Rings, and certain candies from Willy Wonka and the Chocolate Factory (a movie slightly before my time). It's like magic when you pack so much goodness into a small package. This is the allure of the formal definition given above.

However, to digest this definition, it is not enough to simply eat it. It takes training and practice to see the meaning of each part of the definition. I'd like to start with an intuitive notion of the limit, and gradually transform it into the precise definition; then, I'll explain how the precise definition gets condensed into the crazy scribble. Each numbered statement below is a more precise rewording of the previous numbered statement; the text in between should help you get from one to the next. We begin with the basic idea of the limit:

1. As x gets close to c, $f(x)$ gets close to L.

   The target here is L. We want to show that $f(x)$ gets close to L. How close does it get?

2. If x is chosen sufficiently close to c, then the value of $f(x)$ will be as close to L as you like.

   "As close as you like" is pretty close! This is saying that no matter how close together you want $f(x)$ and L to be, your demands can be satisfied by choosing x "sufficiently close" to c. The mathematical way of saying that two things are close together is: the distance between them is less than some number. Usually you say how close $f(x)$ must be to L by specifying a number ε (the Greek letter epsilon), and I say how close x is to c by specifying a number δ (the Greek letter delta). The smaller ε is, the closer $f(x)$ is to L; the smaller δ is, the closer x is to c. Proofs involving the formal definition of limit are often called "Epsilon–Delta Proofs".

3. If you choose a value of ε, then I will tell you a value of δ. The significance of my choice of δ is that, so long as x is within δ of c, $f(x)$ will be within ε of L.

   Recall formula 3 from page 0-9, which tells us that the distance from $f(x)$ to L is $|f(x) - L|$. We represent the statement "$f(x)$ is within ε of L" by saying $|f(x) - L| < \varepsilon$. Likewise, when we talk about x being close to c, we say that $|x - c| < \delta$.

4. For any choice of ε, there exists a number δ, such that: if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

   This is *almost* the precise definition, but it needs a little polishing. First of all, the numbers ε and δ have to be positive. This might be clear if you know that they represent distances, but our precise definition needs to be explicit about stuff like that. Second, the limit of a function does NOT depend in any way upon the value of $f(c)$, so we want to consider all x values within δ of c, *except c itself.* The sneaky shorthand for that is to say that the distance from x to c is a *positive* number less than δ.

5. For any $\varepsilon > 0$, there exists $\delta > 0$ such that: if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This is the precise definition of the limit, but not the version of it that caught my attention so well. We mathematicians have a little shorthand for a few things we say often. For example, "for example" is written "e.g." (from the Latin *exempli gratia* or something like that). "For all" is written "$\forall$". "There exists" is written "$\exists$". "Such that" is often written "s.t.", although I prefer "$\ni$". "Implies" is written "$\Rightarrow$", but that requires a little more explanation—if A and B are statements, then the following sentences are all equivalent:

- If A, then B.
- A implies B.
- $A \Rightarrow B$.

This gives us the condensed version of the precise definition of the limit:

$$\forall\varepsilon > 0 \ \ \exists\delta > 0 \ \ni [0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon] .$$

And now I will use this precise definition to *prove* that the limit as x approaches 2 of $3x - 1$ is 5.

Proof: Let $\varepsilon > 0$ be given. (This is the part where you specify how close you want $3x - 1$ to be to 5. In response, I give a value for δ.) Let $\delta = \varepsilon/3$. Now, if $0 < |x - 2| < \delta$, it follows that $|x - 2| < \delta$. Therefore, $|(3x - 1) - (5)| = |3x - 6| = 3|x - 2| < 3\delta = \varepsilon$.

Basically, my argument allows you to specify any positive number ε you like (this is the "vertical tolerance", or how close the function $f(x) = 3x - 1$ is expected to be to 5). It produces a response, $\delta = \varepsilon/3$ (the "horizontal tolerance"), and then proves that whenever $0 < |x - 2| < \delta$, $|f(x) - 5| < \varepsilon$. Here c has been replaced by 2, and L by 5. Proofs of this kind are generally reviled by analysis students everywhere, although they get less difficult (and more interesting) as you get used to them. Epsilon–Delta proofs for lines become automatic, but for everything else they stay a little involved. The derivative is very helpful in creating these proofs, but I won't say how.

I'd also like to give you the formal definitions of some limits when infinity gets into the mix. It's not like x can get within a certain fixed distance of infinity, so it doesn't make sense to say $|x - \infty| < \delta$. To say "x is close to ∞" is to say "x is large". More formally: $x > M$, or "x exceeds whatever bound is suggested". The bigger M is, the closer x is to ∞. So, for example, the precise definition of

$$\lim_{x\to\infty} f(x) = L$$

is "for any $\varepsilon > 0$, there exists a number M such that: if $x > M$, then $|f(x) - L| < \varepsilon$". Or, since I'm in the mood for some concentrated definitions:

$$\forall\varepsilon > 0 \ \ \exists M \ni [x > M \Rightarrow |f(x) - L| < \varepsilon] .$$

We can also define

$$\lim_{x\to c} f(x) = \infty$$

as

$$\forall N \ \ \exists\delta > 0 \ni [0 < |x - c| < \delta \Rightarrow f(x) > N] .$$

Notice that in this case, $f(x)$ is getting big—and rather than seeing anything like $|f(x) - L| < \varepsilon$, we have $f(x) > N$. Finally, I'll give the precise definition for

$$\lim_{x\to c^-} f(x) = -\infty.$$

Here x approaches c from the left, so I don't care what happens to the right of c; in other words, I may assume $x < c$. Furthermore, if $f(x)$ is going to $-\infty$, that means it ducks under any proposed bound; this explains the very last bit of the definition. The definition is

$$\forall N \;\; \exists \delta > 0 \;\ni [c - \delta < x < c \Rightarrow f(x) < N] \,.$$

I've never encountered this particular definition before, but I know that it's right because math is a language I speak. When I was younger, I memorized interesting strings of symbols the same way I memorized vocabulary words in a foreign language—but at some point, there are too many words to simply memorize them; you have to develop a feel for what they mean. The same thing applies to trig tables, derivatives, and even rules for adding or multiplying numbers. But I'm on the edge of a soliloquy now, so I'll just stop.